The distance from congruence distributivity to near unanimity

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joint work with Alexandr Kazda and Jakub Bulín

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The distance from CD to NU

CD . . . Algebras in a congruence distributive variety
NU . . . Algebras with a near unanimity term operation

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Theorem (Berman, Idziak, Marković, McKenzie, Valeriote, Willard'10; Marković, McKenzie'08; Kearnes, Szendrei'12)

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FinRel . . . Finite, finitely related algebras

Theorem (B'13, Zhuk)

$NU = CD \cap FinRel$

[picture]

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Mentioned results: Talk about *specific* classes of algebras

This talk: For finite algebras, we give more general results which are

- talking about *all* algebras and
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Other examples of "wrong" results:

- The problem "given $\mathbf{A}$ decide whether $\mathbf{A} \in NU$" is decidable Maróti'09 (compare: $\mathbf{A} \in CD$ is obviously decidable)
- The problem "given $\mathbb{A}$ decide whether $\mathrm{Pol}(\mathbb{A}) \in NU$" is decidable B (compare: $\mathrm{Pol}(\mathbb{A}) \in CD$ is obviously decidable)
- Relational characterization of NU Baker-Pixley'75
- Relational characterization of CD Freese, Valeriote'09
- Directed Jónsson terms Kozik

Outline

- FinRel, CD, NU, Cube
- Absorption and (directed) Jónsson absorption
- Better versions of some results

FinRel, CD, NU, Cube

FinRel - Finitely related algebras

$\mathbf{A}$ ... finite algebra

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$\mathrm{Inv}(\mathbf{A}) = SP(\mathbf{A})$ ... subpowers = invariant relations
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Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\mathrm{Pol}(\mathrm{Inv}(\mathbf{A})) = \mathrm{Clo}(\mathbf{A}), \qquad \mathrm{Inv}(\mathrm{Pol}(\mathbb{A})) = \mathrm{RelClo}(\mathbb{A})$

$\Rightarrow$ For every $\mathbf{A}$ there exists $\mathbb{A}$ such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

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$\mathrm{Pol}(\mathrm{Inv}(\mathbf{A})) = \mathrm{Clo}(\mathbf{A})$, $\mathrm{Inv}(\mathrm{Pol}(\mathbb{A})) = \mathrm{RelClo}(\mathbb{A})$

$\Rightarrow$ For every **A** there exists $\mathbb{A}$ such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Definition

A is finitely related, if $\exists\mathbb{A}$ with finitely many relations such that $\mathrm{Clo}\,\mathbf{A} = \mathrm{Pol}(\mathbb{A})$.

CD – Congruence distributivity

$\mathbf{A} \in CD$ if $\quad \forall \mathbf{X} \in \mathsf{HSP}(\mathbf{A}) \quad \forall \beta, \gamma, \delta \in \mathsf{Con}(\mathbf{X})$

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  - Take $\mathbf{B} = \mathbf{C} = \mathbf{D} = \mathbf{F}(x,y)$
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  - Take $\mathbf{B} = \mathbf{C} = \mathbf{D} = \mathbf{F}(x, y)$
  - and $X = \langle xxx, xyy, yxy \rangle$, $b = x$.
  - We get ternary terms $p_0, \ldots, p_n$ such that

    $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
    $p_{2i}(x, y, y) \approx p_{2i+1}(x, y, y), \quad p_{2i+1}(x, x, y) \approx p_{2i+2}(x, x, y)$
    $p_i(x, y, x) \approx x$

CD cntd.

- CD = directed Jónsson terms Kozik

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  - We have $X = \langle xxx, xyy, yxy \rangle \leq \mathbf{F}(x, y)^3$ with bad connectivity properties
  - $\mathbf{F}(x, y) \leq \mathbf{A}^{A^2}$
  - We have $X \leq \mathbf{A}^k \times \mathbf{A}^k \times \mathbf{A}^k$ with bad connectivity properties

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  - We have $X \leq \mathbf{A}^k \times \mathbf{A}^k \times \mathbf{A}^k$ with bad connectivity properties
  - If $\mathbf{A}$ idempotent we can find bad relation for $k = 1$
    Freese, Valeriote'09; B, Kazda
  - This can be used for effectively deciding CD

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- $NU \subseteq CD$
  - $p_i(x, y, z) = t(x, \dots, x, y, z, \dots, z)$ are directed Jónsson terms
  - different proof later

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- $\neg NU$ = bad relations Baker, Pixley'75
- **Theorem:** $CD \cap FinRel = NU$ B'13, Zhuk

Cube – Cube term

$\mathbf{A} \in \mathit{Cube}$ if $\mathbf{A}$ has a term operation t satisfying some identities of the form

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- **Theorem:** $\mathit{Cube} \subseteq \mathit{CM}$ BIMMVW'10, KS'12

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- **Theorem:** $\mathit{Cube} = \mathit{CM} \cap \mathit{FinRel}$ B

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- **Theorem:** $CD + Cube \Leftrightarrow NU$ BIMMVW'10, MM'08, KS'12

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- **A** $\in$ *Cube* iff its full idempotent reduct is in *Cube*
- For idempotent **A**: $\neg Cube$ = bad relations
  Marković, Maróti, McKenzie'12; B, Kozik, Stanovský

Cube cntd.

Definition

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- **Easy fact:** $\forall n \;\; \exists$ blocker $\Rightarrow \mathbf{A} \notin \mathit{Cube}$

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- **Easy fact:** $\forall n \ \exists$ blocker $\Rightarrow \mathbf{A} \notin \mathit{Cube}$
- **Theorem:** $\forall n \ \exists$ blocker $\Leftrightarrow \mathbf{A} \notin \mathit{Cube}$ (MMM'12, BKS)

Short summary

For CD/NU/Cube:

- Characterized by terms
- Properties of the full idempotent reduct
- For idempotent finite algebras, negation is equivalent to the existence of a bad relation
- For Cube the bad relations are cube term blockers (of every arity)

Absorption and Jónsson absorption

"Give up your selfishness, and you shall find peace; like water mingling with water, you shall merge in absorption."

Sri Guru Granth Sahib

Absorption (generalizes NU)

Definition

Let $\mathbf{A}$ be idempotent. Then B absorbs $\mathbf{A}$, written $B \lhd \mathbf{A}$, if $B \leq \mathbf{A}$ and $\exists$ idempotent term t (arity ≥ 2) such that $t(B, B, \ldots, B, A, B, B \ldots, B) \subseteq B$ (for any position of A)

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- **Fact:** $\mathbf{A} \in NU$ iff $\forall a \in A \ \ \{a\} \triangleleft \mathbf{A}$

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- if A is non-idempotent, we define $B \lhd \mathbf{A}$ if B absorbs the idempotent reduct of $\mathbf{A}$
- **Fact:** $\mathbf{A} \in NU$ iff $\forall a \in A \;\; \{a\} \lhd \mathbf{A}$
- $\mathbf{A} \in NU$ is a strong condition,
  having a proper absorbing subuniverse is quite weak:
  For a finite idempotent $\mathbf{A}$ in a variety omitting type 1:
  - **Theorem:** If β, γ is a pair of non-permuting congruences, $\beta \vee \gamma = 1$, then $\mathbf{A}$ has a proper absorbing subuniverse B, Kozik'12
  - **Theorem:** If no subalgebra of $\mathbf{A}$ has a proper absorbing subuniverse then $\mathbf{A}$ has a Maltsev term B, Kozik, Stanovský
  - **Corollary** (Hobby, McKenzie'88): $\mathbf{A}$ Abelian $\Rightarrow$ $\mathbf{A}$ affine

Absorption cntd.

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- For idempotent algebras $\neg(B \triangleleft \mathbf{A}) \Leftrightarrow$ bad relations

Absorption cntd. – bad relations

Definition

Let $B \leq \mathbf{A}$, $\mathbf{A}$ idempotent. Then $R \leq \mathbf{A}^n$ is a *B-absorption blocker* if $R \cap B^n = \emptyset$ and each projection to $(n-1)$-coordinates intersects B^{n-1}. ($n \geq 2$)

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- **Less obvious:** $\mathbf{A}$ has a B-absorption blocker of arity n iff B does not absorb $\mathbf{A}$ via a term of arity n B, Bulín
- **Consequence:** B does not absorb $\mathbf{A}$ iff $\mathbf{A}$ has a B-absorption blocker of every arity (≥ 2).

Jónsson absorption (generalizes CD)

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- Jónsson absorption absorbs connectivity
- Relational characterization of $\neg(B \triangleleft_j \mathbf{A})$ similar to CD (by Freese, Valeriote) B, Kazda

Original question: How far is NU from CD?

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Better question: How far is absorption from Jónsson absorption?

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Both absorptions absorb connectivity
(absorption sometimes in a nicer way).

No other property was ever used.

$\Rightarrow$ shouldn't be too far...

Better versions of some results

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Theorem (B, Bulín; Kozik)

Let $\mathbf{A}$ be finitely related, $B \leq \mathbf{A}$. Then $B \triangleleft_j \mathbf{A}$ implies $B \triangleleft \mathbf{A}$.

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- Proved using Zhuk's technique
- Alternative approach: **see Kozik's talk**

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- Decidable, for the same reason
- Matching lower bound?

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- **Observe:** if $C \cap B = \emptyset$ and $D \cap B \neq \emptyset$, then the cube term blocker is a B-absorption blocker

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Different formulation: If $B \trianglelefteq_j \mathbf{A}$ but $B \not\trianglelefteq \mathbf{A}$ then there is a very special B-absorption blocker (namely a cube term blocker) of every arity.

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- $S \triangleleft_j R$, R connected $\Rightarrow$ S connected $\Rightarrow$ $11 \in S$.

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