Fully Invariant and Verbal Congruences

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Verbal Congruences

$\mathbf{A}$ an algebra, Σ a set of equations

$$\lambda_\Sigma^{\mathbf{A}} = \mathrm{Cg}^{\mathbf{A}}\Big\{\big(s(a_1,\dots,a_n),\ t(a_1,\dots,a_n)\big) : (s \approx t) \in \Sigma,\ a_1,\dots,a_n \in A\Big\}$$

$\lambda_\Sigma^{\mathbf{A}}$ is the smallest congruence θ such that $\mathbf{A}/\theta \models \Sigma$

$\lambda_\Sigma^{\mathbf{A}}$ is the *verbal congruence induced by* Σ

Alternate definition

$\mathcal{V}$ a variety (same similarity type as **A**)
$\Lambda^{\mathbf{A}}_{\mathcal{V}} = \{\, \theta \in \mathrm{Con}(\mathbf{A}) : \mathbf{A}/\theta \in \mathcal{V} \,\}$
$\lambda^{\mathbf{A}}_{\mathcal{V}} = \bigcap \Lambda^{\mathbf{A}}_{\mathcal{V}}$

Easy to see:
- $\mathbf{A}/\lambda_{\mathcal{V}} \in \mathcal{V}$
- If $\mathcal{V} = \mathrm{Mod}(\Sigma)$ then $\lambda_{\mathcal{V}} = \lambda_{\Sigma}$

Theorem

*θ is verbal on **A** iff*

$$\mathbf{A}/\psi \in \mathrm{Var}(\mathbf{A}/\theta) \implies \psi \geq \theta.$$

Examples from Group Theory

Suppose $\Sigma = \{xy \approx yx\}$

On any group **A**, $\lambda_{\Sigma}^{\mathbf{A}} = \operatorname{Cg}\{\,(ab, ba) : a, b \in A\,\}$
corresponds to $\mathbf{A}' = \operatorname{Nml}\{\,[a, b] : a, b \in A\,\} = [A, A]$

$\mathbf{A}/\mathbf{A}'$ is the largest Abelian homomorphic image of **A**.

$\Theta_n = \{x^n \approx e\}$

$\lambda^{\mathbf{A}}_{\Theta_n}$ corresponds to $\mathrm{Nml}\{\, a^n : a \in A \,\}$

$\mathbf{A}/\lambda_{\Theta_n}$ is the largest homomorphic image of **A** of exponent n.

Note: $\lambda_\Sigma \leq \lambda_{\Theta_2}$ since every group of exponent 2 is Abelian

Fully Invariant Congruences

$\mathrm{End}(\mathbf{A})$ = endomorphism monoid of $\mathbf{A}$

A congruence θ is *fully invariant* if
$\forall f \in \mathrm{End}(\mathbf{A}) \quad (a, b) \in \theta \implies \big(f(a),\, f(b)\big) \in \theta$

Theorem

Every verbal congruence is fully invariant

Converse is false

Example: Let p be prime,
$\mathbf{A}$ the Abelian group $\langle a, b \mid pa = p^2 b = 0 \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$.

$$S = \{\, x \in A : px = 0 \,\} \supsetneq T = \{\, px : x \in A \,\}$$

S is fully invariant but
$\mathbf{A}/T \in \mathrm{Var}(\mathbf{A}/S)$ so S not verbal

General Question

Find conditions under which

$$\text{fully invariant} \implies \text{verbal}$$

- for a congruence
- for an algebra
- for a variety.

An algebra is called *verbose* if every fully invariant congruence is verbal.
A variety is *verbose* if every member is verbose.

Verbose Varieties

Theorem

- *A variety of Abelian groups is verbose if and only if it is of square-free exponent.*
- *The lattice varieties* $\mathbf{V}(\mathbf{M}_n, \mathbf{N}_5)$ *are verbose. The variety* $\mathbf{V}(\mathbf{M}_{3,3})$ *is not verbose.*

Theorem

Let $\mathcal{V}$ be a finitely generated discriminator variety. Then $\mathcal{V}_{fin}$ is verbose.

Proof.

$\mathbf{A} \in \mathcal{V}_{\text{fin}} \implies \mathbf{A} \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ all simple.
$\eta_i = \ker(\mathbf{A} \twoheadrightarrow \mathbf{A}_i)$.
Suppose $\theta \in \text{Con}(\mathbf{A})$ not verbal. Then
$\exists i, j \;\; \theta \leq \eta_i, \; \theta \nleq \eta_j, \; \mathbf{A}_j \overset{h}{\rightarrowtail} \mathbf{A}_i$

Define $e(\mathbf{x}) = (x_1, \ldots, x_{i-1}, h(x_j), x_{i+1}, \ldots, x_n)$.
Then $e \in \text{End}(\mathbf{A})$ but
$(\mathbf{a}, \mathbf{b}) \in \theta - \eta_j \implies \big(e(\mathbf{a}), e(\mathbf{b})\big) \notin \theta$. □

Can this argument be extended to infinite algebras?

Need a representation that is “almost as good” as direct product

Answer: NU-duality

A finite algebra $\mathbf{M}$ is *subalgebra-primal* if $\mathrm{Clo}(\mathbf{M}) = \mathrm{Pol}(\mathrm{Sub}(\mathbf{M}))$.

$\mathbf{M}$ is quasi-primal and rigid

Assume $\mathcal{V} = \mathrm{Var}(\mathbf{M})$, $\mathbf{M}$ subalgebra-primal

$\mathbf{A} \in \mathcal{V} \implies \mathbf{A} \rightsquigarrow \langle X, T, S \rangle$, T a Boolean Topology on X, S a family of closed subspaces

Instead of constructing endomorphism $\mathbf{A} \xrightarrow{e} \mathbf{A}$
build continuous map $X \xleftarrow{\widehat{e}} X$ preserving S

We can imitate the finite argument to show $\mathcal{V}$ is verbose

Should be true for finitely generated discriminator variety.

Need the continuous functions $X \xleftarrow{\widehat{e}} X$ to preserve the "inner automorphisms" of $\mathbf{A}$.

Middle ground: Can show that $\mathbf{V}(\mathbf{F})$ is verbal where $\mathbf{F}$ is a finite field.