Evolutionary Computation: Term to Term Operation Continuity

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For example, **P** is primal:

$$\mathbf{P} := \langle \{0, 1, 2, 3, 4\}, *\rangle$$

$*$	0	1	2	3	4
0	1	1	0	2	0
1	4	2	0	3	3
2	1	0	3	2	4
3	0	4	3	4	2
4	2	0	1	4	0

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2008 Hummie Award from the ACM.

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$$
\begin{aligned}
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& x*x*xxxx***xxx*xx*zx****x**xzx**xx**x*x*xxx*x*x* \\
& *xxxxxxx******xzxx*x***xxxx***xx*xx**xxx*xx*zx***x* \\
& *xx*xx**xxx*xx**x**xxxxx****zxxx***xxx**xx**x*xxxx* \\
& **x*x*xxxxx*x**x*x***xxxzx***xxx***x**xx*x*x*x* \\
& xxxzxxx*****x**xxxx***xx**xx*xxz*x*xz**x**xxx**xxx*x* \\
& *xx*x**xz*xx**x*zx*x*x*xzzxx**x**x**zxx**x*xxxzxzz* \\
& ****x**yyxx**x**xzyz**xx***xxxzzxx*x*****y**xxx*xx* \\
& **zzx**xx*xy**y*zy*x*x*zxx**zxx**x****xx*x*z***** \\
& yy*z*xx*y**x**xxxxx*x*x**x**x*****x*xxy*xx***x**xyzx* \\
& xx***z*xx**x***yyz**zy****x**xzzxxx**zy****x***zy*x*y* \\
& y*x*x*****zx*xxxx***x******zzz***yyx*yx****xx*x***** \\
& xyy*xx**x*x********xxx*zx**x***********************
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Discriminator term for **P** — found by EC in 53 seconds!

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x*x*xxxx***xxx*xx*zx****x**xzx**xx**x*x*xxx*x*x*
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****x**yyxx**x**xzyz**xx***xxxzzxx*x******y**xxx*xx*
zzxxx*xy**y*zy*x*x*zxx**zxx**x****xx*x*z*****
yy*z*xx*y**x**xxxxx*x*x***x*****x*xxy*xx***x**xyzx*
xx***z*xx**x***yyz**zy****xzzxxx**zy****x***zy*x*y*
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Under HD, the space G^{G^k} is a metric space.

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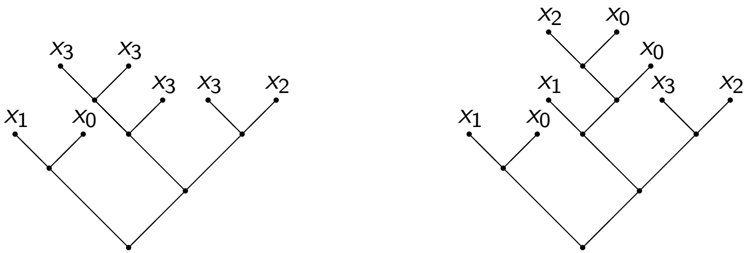

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Figure: Distance measure on the term space $\mathcal{T}$.

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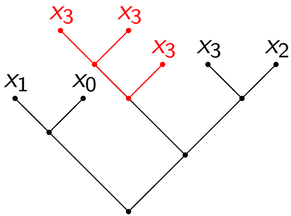

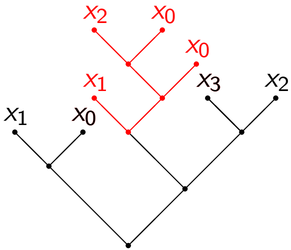

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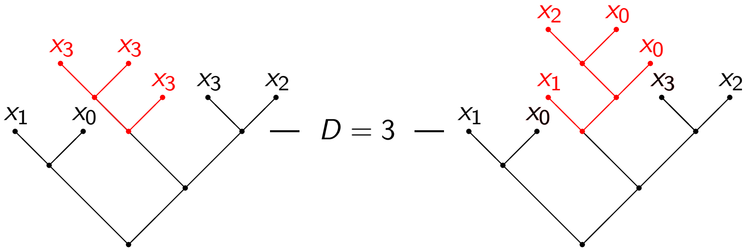

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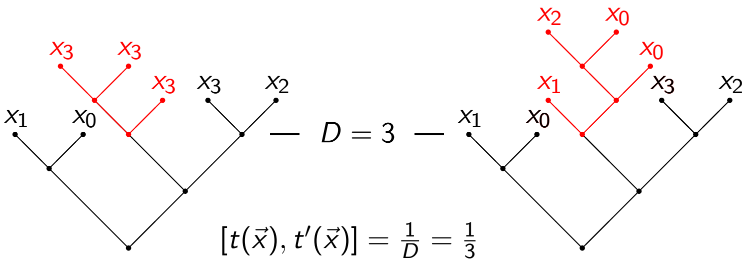

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$$\mathcal{M}_H := \{M \mid t(\vec{x}, u(\vec{x})) \in \mathcal{T}_H\}, \text{ a finite set}$$

If D is the depth of $\Diamond$ in $t(\vec{x}, \Diamond)$, then the **change** in the term $t(\vec{x}, u(\vec{x}))$ resulting from applying $M = \langle t(\vec{x}, \Diamond), u(\vec{x}), v(\vec{x}) \rangle$ is

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1. $\sigma \neq \varnothing$,
2. $(a, b) \in \sigma$ implies $(b, a) \in \sigma$,
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Test for Separating Relations: Either shows there are no separating relations or produces one. E.g., **P** has none.

Let $\vec{d} \in G^k$ and let $a \in G$. For $H \in \mathbb{N}$ define

$$\beta_{\vec{d},a}(H) := \text{Prob}\langle t(\vec{d}) = a \mid t \in \mathfrak{T}_H \rangle.$$

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Definition. The groupoid **G** is **asymptotically complete** if, for each $k \in \mathbb{N}$, each $\vec{d} \in G^k$ and each $a \in \text{sg}\{d_0, d_1, \ldots, d_{k-1}\}$, the sequence $\beta_{\vec{d},a}$ is eventually bounded away from 0.

Test for Asymptotic Completeness.

P distributions with $k = 10$ and $\vec{d} = (4, 3, 0, 3, 1, 3, 3, 0, 2, 3)$.

H	$\beta_{\vec{d},0}(H)$	$\beta_{\vec{d},1}(H)$	$\beta_{\vec{d},2}(H)$	$\beta_{\vec{d},3}(H)$	$\beta_{\vec{d},4}(H)$

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1	0.2	0.1	0.1	0.5	0.1

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2	0.172727	0.100000	0.118182	0.318182	0.290909

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20	0.292581	0.236024	0.173479	0.121485	0.176431
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Experimental evidence that **P** is asymptotically complete.

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G: a finite asymptotically complete groupoid with subgroupoids having no separating relations. Let $n := \|G\|$.

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- If $(a, b) \in \sigma$ and $c \in K$, then $(ca, cb), (ac, bc) \in \sigma$.

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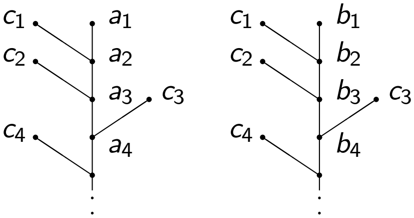

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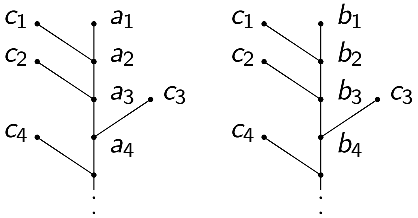

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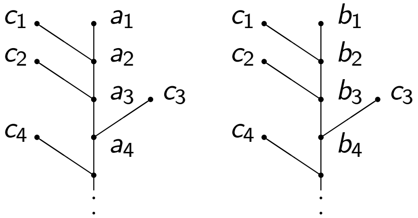

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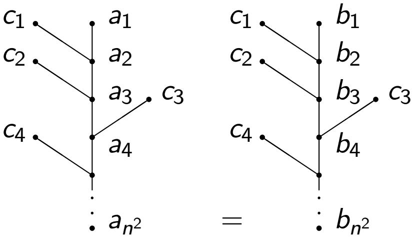
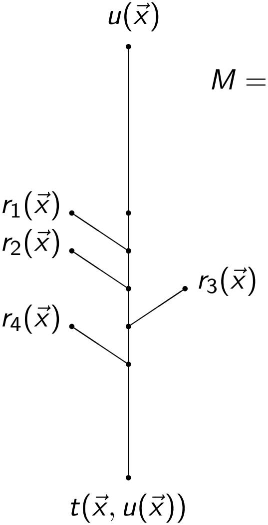
No separating relations implies there is $m \in \mathbb{N}$ such that $a_m = b_m$ with $m \leq n^2$. Thus

$a_{n^2} = b_{n^2}$.



$$M = (t(\vec{x}, \Diamond), u(\vec{x}), v(\vec{x}))$$



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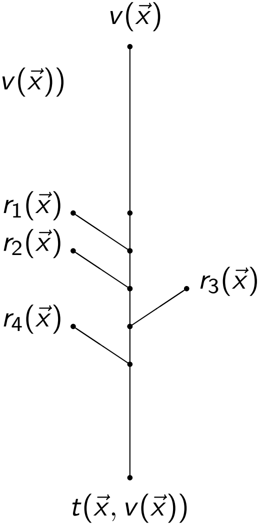

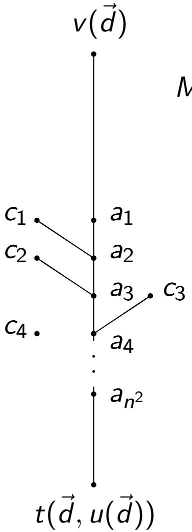

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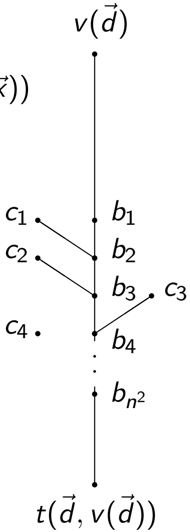
. . . .

$\downarrow$

All values in **K**.



Thanks for listening!

— DC

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