Location of annihilators of $C(X)$ in coherence classes via lattices

Themba Dube

Department of Mathematical Sciences
University of South Africa

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Let A be a commutative ring with identity, and let I be an ideal of A. The pure part of I is the ideal

$$mI = \{a \in I \mid a = ab \text{ for some } b \in I\}.$$

In the article

T.R. Jenkins and J.D. McKnight, Jr.
Coherence classes of ideals in rings of continuous functions
Indag. Math. **24** (1962), 299-306.

the authors define a relation $\equiv$ by

$$I \equiv J \qquad \Longleftrightarrow \qquad mI = mJ.$$

1. A denotes a normal bounded distributive lattice, which we assume to be nontrivial in the sense that $0 \neq 1$.

2. For any $a, b \in A$ we write $a \prec b$ (read: a is rather below b) if there is an $s \in A$ such that

$$a \wedge s = 0 \quad \text{and} \quad s \vee b = 1.$$

3. For any $a \in A$, we set

$$r(a) = \{x \in A \mid x \prec a\}.$$

Definition

Let I be an ideal of A. We define the pure part of I, denoted ϱI, to be the ideal

$$\varrho I = \{a \in A \mid a \prec b \text{ for some } b \in I\}.$$

Thus,

$$\varrho I = \bigcup\{r(u) \mid u \in I\} = \bigvee_{\mathfrak{J}A}\{r(u) \mid u \in I\}.$$

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Observe that ϱI is an ideal of A such that:

1. $\varrho I \subseteq I$,
2. ϱI is a pure ideal, i.e., $\varrho(\varrho I) = \varrho I$.

In analogy with lattice-ordered groups, if I is an ideal of A, we define the polar of I to be the ideal

$$I^{\perp} = \{a \in A \mid a \wedge u = 0 \text{ for each } u \in I\}.$$

Observe that $I^{\perp}$ is actually the pseudocomplement of I in the frame $\mathfrak{J}A$.

For any $a \in A$, we abbreviate $(\downarrow a)^{\perp}$ as $a^{\perp}$.

Lemma

For any ideal I of A, $\varrho I = \{a \in A \mid I \vee a^{\perp} = \top\}$.

Definition

For any ideal I of A we define $\overline{I}$ to be the ideal

$$\overline{I} = \bigcap \{M \in \mathrm{Max}(A) \mid M \supseteq I\}.$$

Membership into this ideal can be described in terms of the pure part as indicated in the following lemma.

Lemma

Let I be an ideal of A. For any $a \in A$ *we have*

$$a \in \overline{I} \iff r(a) \subseteq \varrho I.$$

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Lemma

Let I be an ideal of A. For any $a \in A$ we have

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Lemma

Let I and J be ideals of A. Then

1. $\varrho I = \varrho \bar{I}$.
2. $\bar{I} = \overline{\varrho I}$.
3. $\varrho I = \varrho J$ *if and only if* $\bar{I} = \bar{J}$.
4. $\mathfrak{M}(I) = \mathfrak{M}(\varrho I)$.

Definition

Let $\equiv$ be the relation on $\mathfrak{J}A$ defined by

$$I \equiv J \iff \varrho I = \varrho J.$$

For any $I \in \mathfrak{J}A$ we denote the equivalence class containing I by $[\![I]\!]$ and call the resulting equivalence classes *coherence classes* of $\mathfrak{J}A$.

Theorem

For any $I \in \mathfrak{J}A$,

$$[\![I]\!] = \{J \in \mathfrak{J}A \mid \varrho I \subseteq J \subseteq \bar{I}\}.$$

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Corollary

The following results about coherence classes of A hold.

1. *An ideal is alone in its coherence class if and only if it is pure and is an intersection of maximal ideals.*
2. *The zero ideal is alone in its coherence class.*
3. *Every coherence class contains at most one maximal ideal.*
4. *A maximal ideal is alone in its coherence class if and only if it is pure.*
5. *Any prime ideal is in the same coherence class as every proper ideal containing it.*
6. *If I is a pure prime ideal of A, then $[\![I]\!]$ consists precisely of all ideals containing I. Furthermore, if A is a regular σ-frame, then $[\![I]\!]$ is a bounded chain.*

Theorem

Let I be a prime ideal of A. Then:

1. *ϱI is a maximal pure ideal of A.*
2. $\bar{I} = \{a \in A \mid r(a) \subseteq I\}$.

Corollary

Every ideal of A is alone in its coherence class if and only if A is a Boolean algebra.

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Theorem

If A is a regular σ-frame, then all members of any coherence class have the same polar.

Theorem

Let A be a regular σ-frame. Then we have the following.

1. *No coherence class contains an ideal and its polar.*
2. *No coherence class contains more than one polar.*
3. *Some coherence classes may fail to contain a polar.*
4. *If a coherence class contains no polar, then it contains no prime ideal.*

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Concerning the third statement in the foregoing proposition, we have that, in fact, if every coherence class of Coz L contains a polar, then L is Boolean. For, given any $a \in L$, the coherence class $[\![\langle a \rangle]\!]$ contains a polar, so that, $a = a^{**}$, implying L is Boolean. The converse fails, as witnessed by the following example.

Example

Let $L = \mathfrak{O}\mathbb{N}$. Then $\mathfrak{J}(\text{Coz}\, L) = \mathfrak{J}L$ is the Stone-Čech compactification of L. Note that $\mathfrak{J}L$ is not Boolean, otherwise, being isomorphic to the topology of some compact Hausdorff space, it would be finite – which it is not. So there is an ideal I in $\mathfrak{J}L$ which is not a pseudocomplement. But pseudocomplements in $\mathfrak{J}L$ are precisely the polars. By the corollary above, I is alone in $[\![I]\!]$, so that this coherence class contains no polar.

Lemma

Let P and Q be ideals of $C(X)$. Then P and Q are in the same $C(X)$-coherence class if and only if $\operatorname{coz}[P]$ and $\operatorname{coz}[Q]$ are in the same $\mathfrak{J}\big(\operatorname{Coz}(\mathfrak{O}X)\big)$-coherence class.

Lemma

Let Q be an ideal of $C(X)$. Then $\operatorname{coz}[\operatorname{Ann}(Q)] = (\operatorname{coz}[Q])^{\perp}$.

Corollary

All ideals of $C(X)$ in any coherence class have the same annihilator.

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All ideals of $C(X)$ in any coherence class have the same annihilator.

Theorem

Let X be a nonempty Tychonoff space.

1. *No coherence class of ideals of $C(X)$ contains an ideal and its annihilator.*
2. *No coherence class contains more than one annihilator.*
3. *If X has an open set which is not closed, then some coherence class contains no annihilator.*
4. *If a coherence class contains no annihilator, then it contains no prime ideal.*

Corollary

Every ideal of $C(X)$ is alone in its coherence class if and only if X is a P-space.

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THANK YOU