# A Syntactic Approach to the Complexity of Linear Idempotent Mal'cev Conditions

Jonah Horowitz

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## Theorem (Freese, Valeriote 2009)

Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ supports a majority term operation if and only if for every $0,1,2,3,4,5 \in A$ there are $6,7,8 \in A$ such that

$$\left(\begin{pmatrix}0\\2\\5\end{pmatrix},\begin{pmatrix}6\\7\\8\end{pmatrix}\right) \in Cg_{\mathbf{A}^3}\left(\begin{pmatrix}0\\2\\5\end{pmatrix},\begin{pmatrix}0\\3\\4\end{pmatrix}\right) \wedge Cg_{\mathbf{A}^3}\left(\begin{pmatrix}0\\2\\5\end{pmatrix},\begin{pmatrix}1\\2\\4\end{pmatrix}\right) \text{ and}$$

$$\left(\begin{pmatrix}6\\7\\8\end{pmatrix},\begin{pmatrix}1\\2\\4\end{pmatrix}\right) \in Cg_{\mathbf{A}^3}\left(\begin{pmatrix}0\\3\\4\end{pmatrix},\begin{pmatrix}1\\2\\4\end{pmatrix}\right) \wedge Cg_{\mathbf{A}^3}\left(\begin{pmatrix}0\\2\\5\end{pmatrix},\begin{pmatrix}1\\2\\4\end{pmatrix}\right).$$

$p$ is a Mal'cev operation on $A$ if for every $x, y \in A$

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### Theorem (Freese, Valeriote 2009)

Let **A** be a finite idempotent algebra. Then **A** supports a Mal'cev term operation if and only if for every $0, 1, 2, 3 \in A$ it is the case that

$$(\binom{1}{3}, \binom{0}{2}) \in Cg_{\mathbf{A}^2}(\binom{0}{2}, \binom{0}{3}) \circ Cg_{\mathbf{A}^2}(\binom{0}{3}, \binom{1}{3}).$$

**Corollary (Freese, Valeriote 2009)**

Whether or not a finite idempotent algebra possesses a majority term can be determined in polynomial time.

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Whether or not a finite idempotent algebra possesses a Mal'cev term can be determined in polynomial time.

Let $p$ be a Mal'cev operation.
Suppose that $p_1$ is almost a Mal'cev operation, but for a few $x, y \in A$ $p_1(y, y, x) \neq x$.
Suppose that $p_2$ is almost a Mal'cev operation, but for a few $x, y \in A$ $p_2(x, y, y) \neq x$.

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Then we can define

$$p'(x, y, z) := p(p_1(x, y, z), p_1(y, y, z), z)$$

$$p''(x, y, z) := p(x, p_2(x, y, y), p_2(x, y, z))$$

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Both $p'$ and $p''$ are Mal'cev operations.
So what?

Suppose that for every $a, b, c, d \in A$ there is an idempotent $p$ such that

$$p(a, b, b) = a \text{ and}$$

$$p(d, d, c) = c.$$

Call $p$ a local Mal'cev operation on $a, b, c, d$.

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Call $p$ a local Mal'cev operation on $a, b, c, d$.
Pick $e, f \in A$ and let $q$ be a local Mal'cev operation on $a, b, e, p(f, f, e)$ and let $q'$ be a local Mal'cev operation on $e, p(e, f, f), c, d$.

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$$p'(x, y, z) := q(p(x, y, z), p(y, y, z), z)$$

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Then $p'$ is a local Mal'cev operation on $a, b, c, d$ and on $a, b, e, f$ and $p''$ is a local Mal'cev operation on $a, b, c, d$ and on $e, f, c, d$.

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Let **A** be a finite idempotent algebra. Then **A** supports a Mal'cev term operation if and only if for every $0, 1, 2, 3 \in A$ it is the case that

$$\binom{0}{2} \in Sg_{\mathbf{A}}\{\binom{0}{3}, \binom{1}{3}, \binom{1}{2}\}.$$

## based on Berman, Idziak, Marcovič, McKenzie, Valeriote, Willard 2010

Let Γ be a set of columns of $x$'s and $y$'s (of height $n$) and let **A** be a finite algebra. If

$$\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} \in Sg_{\mathbf{F}_{V(\mathbf{A})}\{x,y\}}\Gamma$$

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then call a term which witnesses this fact a Γ-special cube term.

So a majority term is a $\{\begin{pmatrix} y \\ x \\ x \end{pmatrix}, \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \begin{pmatrix} x \\ x \\ y \end{pmatrix}\}$-special cube term,

and a Mal'cev term is a $\{\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}\}$-special cube term.

Let $\Gamma$ be a set of columns of $x$'s and $y$'s (of height $n$), let $a, b \in A$, let $i < n$ and let $p$ be an operation on $A$ whose variables are indexed by $\Gamma$. Define $\gamma_i : A^2 \to A^\Gamma$ to be the function where

$$\gamma_i(a,b)(C) := \begin{cases} a & \text{if the } i\text{th element of } C \text{ is } x \\ b & \text{otherwise} \end{cases}$$

Say that $p$ is a local $\Gamma$-special cube operation on $(a, b, i)$ if $p(\gamma_i(a, b)) = a$. Given any subset $S$ of $A^2 \times \{0, \ldots, n-1\}$, say that $p$ is a local $\Gamma$-special cube operation on $S$ if $p$ is a local $\Gamma$-special cube operation on $(a, b, i)$ for every $(a, b, i) \in S$.

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## Theorem

Let $\Gamma$ be an order ideal in the $n$th power of the semilattice $x < y$ and let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ supports a $\Gamma$-special cube term if and only if there is a local $\Gamma$-special cube operation on $S$ for every $S \subseteq A^2 \times \{0, \ldots, |\Gamma| - 1\}$ with $|S| = |\Gamma|$.

Fix $\Gamma = (C_0, \ldots, C_{n-1})$ and $A$.
Assume that for some $k \geq n$ $A$ supports local $\Gamma$-special cube operations on all sets of size $k$. It suffices to show for an arbitrary $S \subseteq A^2 \times \{0, \ldots, n-1\}$ with $|S| = k + 1$, that $A$ supports a local $\Gamma$-special cube operation on $S$.
Choose $(a, b, i) \in S$ with $|S| \cap (A^2 \times \{i\})| > 1$ and define

$$T := S \setminus \{(a, b, i)\}$$

$$R := S \setminus (A^2 \times \{i\}) \cup \{(a, p_T(\gamma_i(a, b)), i)\}$$

where $p_T$ is the local $\Gamma$-special cube operation on $T$. For each $j < n$ define

$$z_j(\overline{x}) := \begin{cases} x_j & \text{if } C_j(i) = x \\ p_T(\overline{x}) & \text{if } C_j \text{ has exactly one } y\text{, at } i \\ p_T(\gamma_i(x_{q_j}, x_j)) & \text{otherwise} \end{cases}$$

where $C_{q_j}$ is the column covered by $C_j$ such that they differ only in position $i$. Then define

$$p_S(\overline{x}) := p_R(\overline{z}(\overline{x})).$$

**Corollary**

Given $k \geq 3$, it is checkable in polynomial time whether or not a finite idempotent algebra supports a $k$-ary near unanimity term.
Given $k \geq 2$, it is checkable in polynomial time whether or not a finite idempotent algebra supports a $k$-edge term.

Note that McKenzie independantly arrived at similar results to this corollary for near unanimity terms and Mal'cev terms using different methods.

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The interesting case: if $i = 0$, define

$$T := S \setminus \{(a, b, 0)\}$$

$$Q := \{(d, c, 1) | (c, d, 2) \in T\} \cup \{(d, c, 2) | (c, d, 1) \in T\} \cup \{(a, p_T(a, b, a), 0)\}$$

$$R := S \setminus (A^2 \times \{0\}) \text{ and define}$$

$$p_S(x, y, z) = p_R(x, p_Q(x, p_T(x, y, z), z), z)$$

**Theorem (Valeriote, 2013)**

A similar construction works for congruence $n$-permutability.

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