A proof that categorical equivalence is decidable

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Let A and B be algebras (not necessarily having similar type).

Question

When there is a categorical equivalence $\varphi : \mathcal{V}(A) \to \mathcal{V}(B)$ such that $\varphi(A) \simeq B$? (This condition is referred as A is categorically equivalent to B.)

Several Answers:

- Characterization by terms (McKenzie 1996).
- Characterization by relational clones (Denecke and Lüders 2001).
- The case of finite groups.
- The case of maximal clones.
- etc.

Topic of this talk is the case of "recursive algebras".

Theorem (Bergman and Berman 1998 [3])

There is an algorithm that

- Input: Finite sets A and B and finite sets of operations F on A and G on B.
- Output: Whether A is categorically equivalent to B.

Theorem (Folklore?)

There is an algorithm that

- Input: Finite sets A and B and finite sets of relations R on A and S on B.
- Output: Whether $(A, \mathrm{Pol}(S))$ is categorically equivalent to $(B, \mathrm{Pol}(T))$.

Remark

There are no algorithms that decide equality of functions determined by codes of algorithms.
It seems there are no algorithms that decide categorical equivalence for any recursive algebras.

Outline of Algorithm

Theorem (I. 2013 [6])

Finite algebras A and B are categorically equivalent iff $\mathrm{Ess}(A) \simeq \mathrm{Ess}(B)$.

1. Calculate essential parts and its generators of input algebras.
2. Compare the cardinality of underlying sets of essential parts.
3. For each bijection, examine definitional equivalence hold or not.

Outline of Talk

1. Definitional Equivalence

Definition

Let A be a finite set.

- Let F be a set of operations on A.
  $\mathrm{Inv}(F) := \{R \subset A^n \mid n \in \mathbb{N}, R \text{ is closed under all } f \in F\}$.
- Let R be a set of relations on A.
  $\mathrm{Pol}(R) := \{f : A^n \to A \mid n \in \mathbb{N}, f \text{ preserves all } r \in R\}$.

Theorem

Let A be a finite set, F be a set of operations on A and g be an operation on A. TFAE.

- g is described by function belongs to F using compositions and projections. ($g \in \langle F \rangle$)
- $g \in \mathrm{Pol}(\mathrm{Inv}(F))$, namely for any r closed under F, r also be closed under g.

Corollary

Let A be a finite set. Then there is an algorithm that

- Input: Finite sets of operations F and G on A.
- Output: Whether $\langle F \rangle = \langle G \rangle$.

Sketch of Proof: If $\langle F \rangle = \langle G \rangle$ is true, then we can conclude $\langle F \rangle = \langle G \rangle$ in finite steps.
If $\langle F \rangle \nsubseteq \langle G \rangle$ is true, then we can find $r \in \mathrm{Inv}(G)$ s.t. r is not closed under some $f \in F$ and can conclude $\langle F \rangle \nsubseteq \langle G \rangle$.

Theorem

Let A be a finite set, R be a set of relations on A and s is a relation on A. Then TFAE.

- s is definable by $\exists, \wedge$ and relations belongs to R. ($s \in [R]$)
- $s \in \mathrm{Inv}(\mathrm{Pol}(R))$.

Corollary

Let A be a finite set. Then there is an algorithm that

- Input: Finite sets of relations R and S on A.
- Output: Whether $[R] = [S]$ (that is, equivalent to $\mathrm{Pol}(R) = \mathrm{Pol}(S)$).

2. Relational Structure Theory

Many definitions in this part.

- Idempotent retract. (Compression.)
- Matrix product. (Constructed algebra from family of idempotent retracts.)
- Cover relation. (No lack of information with respect to category structure.)
- Smallness of family of idempotent retracts.
- Essential part
  $=$ matrix product of the minimum family of idempotent retracts that has no lack of information.

Definition (Kearnes 2001 [1])

Let A be an algebra.
· $e \in \mathrm{Clo}_1(A)$ is said to be idempotent if $e^2 = e$.
$\mathbf{E}(A)$:= the set of all idempotent term operations.
· If $e \in \mathbf{E}(A)$, an idempotent retract of A by e, denoted by $A|_{e(A)}$ or simply $e(A)$, is an algebra such that

- The underlying set is $e(A)$.
- $\mathrm{Clo}_m(A|_{e(A)}) := \{e \circ f \mid f \in \mathrm{Clo}_m(A)\}$.

The structure of $A|_{e(A)}$ is determined by the set $e(A) = \{x \in A \mid e(x) = x\}$ (independent of e itself).

Remark

$X \mapsto e(X)$ is a functor from $\mathcal{V}(A)$ to $\mathcal{V}(e(A))$.

Definition ([1])

Let A be an algebra. $e_1, \ldots, e_l \in \mathbf{E}(A)$. $e_1(A) \boxtimes \cdots \boxtimes e_l(A)$ (called matrix product) is an algebra such that
- the underlying set is $e_1(A) \times \cdots \times e_l(A)$.
- $\mathrm{Clo}_m(e_1(A) \boxtimes \cdots \boxtimes e_l(A))$ is a set of tuples $(e_1 t_1, \cdots, e_l t_l)$ where $t_i \in \mathrm{Clo}_{lm}(A)$.

Remark

$X \mapsto e_1(X) \boxtimes \cdots \boxtimes e_l(X)$ is a functor from $\mathcal{V}(A)$ to $\mathcal{V}(e_1(A) \boxtimes \cdots \boxtimes e_l(A))$.

Definition

Let $e_1, \ldots, e_n, e \in \mathbf{E}(A)$.
$\{e_1(A), \ldots, e_n(A)\}$ covers $e(A)$ if

$$\exists t, f_1, \cdots, f_l \in \mathrm{Clo}(A)\,;\, t(e_{i_1}f_1(x), \cdots, e_{i_l}f_l(x)) = e(x).$$

This case $e(A)$ is "embedded" in $e_{i_1}(A) \times \cdots \times e_{i_l}(A)$ by term operations. That is, idempotent retract

$$(e_{i_1}f_1, \cdots, e_{i_l}f_l) \circ t(e_{i_1}(A) \boxtimes \cdots \boxtimes e_{i_l}(A))$$

is isomorphic to $e(A)$.

$$\begin{array}{rccc} & A & \text{``}\subset\text{''} & e_{i_1}(A) \boxtimes \cdots \boxtimes e_{i_l}(A) \\ \text{``Embedding''}: & x & \longmapsto & (e_{i_1}f_1(x), \ldots, e_{i_l}f_l(x)) \\ \text{``Retraction''}: & t(x_1, \cdots, x_l) & \longleftarrow & (x_1, \cdots, x_l) \end{array}$$

Proposition ([1])

Let $\mathcal{A} = (A, C)$ be a finite algebra and $e_1(A), \ldots, e_n(A)$ be idempotent retracts of $\mathcal{A}$. Then TFAE.

- $\{e_1(A), \ldots, e_n(A)\}$ covers A.
- For $n \in \mathbb{N}$, $S, T \in \mathrm{Inv}_n(C)$, the following implication holds:
$$S\restriction_{e_i(A)} = T\restriction_{e_i(A)} \text{ for all } i \in \{1, \ldots, n\}$$
$$\Rightarrow S = T.$$
- $X \mapsto e_1(X) \boxtimes \cdots \boxtimes e_n(X)$ is a categorical equivalence $\mathcal{V}(A) \to \mathcal{V}(e_1(A) \boxtimes \cdots \boxtimes e_n(A))$.

Definition ([1])

A cover $\mathcal{U} = \{e_1(A), \ldots, e_n(A)\}$ of A is minimum if $\mathcal{U}$ has no proper refinement and $\mathcal{U} \setminus \{e_{i_0}(A)\}$ does not cover A for $i_0 = 1, \cdots, n$.

Definition ([1])

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be covers of A. $\mathcal{U}_1$ is a refinement of $\mathcal{U}_2$ if for all $U_1 \in \mathcal{U}_1$ there exists $U_2 \in \mathcal{U}_2$ such that $U_1 \subseteq U_2$.
An refinement $\mathcal{U}_1$ of $\mathcal{U}_2$ is said proper if $\mathcal{U}_2$ is not refinement of $\mathcal{U}_1$.

Definition/Theorem ([1], Behrisch 2009 [2], [6])

Let A be a finite algebra.

- A has unique minimum cover "up to isomorphism".
- $\mathrm{Ess}(A)$ denotes the matrix product of minimum cover and is said essential part of A.
- A has unique essential part "up to isomorphism".

Theorem ([6])

Let A and B be finite algebras. TFAE.

- A and B are categorically equivalent to each other.
- $\mathrm{Ess}(A) \simeq \mathrm{Ess}(B)$.

3. The Algorithm

Outline of the algorithm

1. Calculate essential parts and its generators of input algebras.
2. Compare the cardinality of underlying sets of essential parts.
3. For each bijection, examine definitional equivalence hold or not.

Calculate essential parts

Lemma

There is algorithms as follows:

- Input: Finitely generated (co-generated) algebra and tuple of idempotent operations $e_1, \ldots, e_n : A \to A$.
- Output: Whether $e_1, \ldots, e_n$ are term operations on A and $\{e_1(A), \ldots, e_n(A)\}$ is minimum cover of A, and if YES, output tuple of the terms satisfying a cover identity. $t(e_{i_1} g_1(x), \ldots, e_{i_l} g_l(x)) = x$.

relational case

If $\{S_1, \ldots, S_N\}$ is a set of generators of the relational clone of A, then $\{\mathrm{Ess}(S_1), \ldots, \mathrm{Ess}(S_n)\}$ is a set of generators of the relational clone of $\mathrm{Ess}(A)$.
Here, $\mathrm{Ess}(S_i)$ is defined as

$$S_i\restriction_{U_1} \times \cdots \times S_i\restriction_{U_n}$$

where

$$\mathrm{Ess}(A) = U_1 \boxtimes \cdots \boxtimes U_n$$

operational case

Suppose $f_1, \ldots, f_n$ is a set of generators of operational clone of A and cover identity

$$t(e_1 g_{11}(x), \ldots, e_1 g_{1M}(x), \ldots, e_N g_{N1}(x), \ldots, e_N g_{NM}(x)) = x$$

holds, where $\{e_1(A), \ldots, e_N(A)\}$ is a minimum cover of A. Then the set of terms determined by the following intuition generates the clone of $\mathrm{Ess}(A)$.

1. Translation to A: $e_{i_0}(A) \hookrightarrow A \simeq \tilde{A} \subseteq \mathrm{Ess}(A)^M$.
2. Copy of generators: $\tilde{f}_s : \tilde{A}^{a_s} \to \tilde{A}$.
3. Recover from A: $\tilde{A}^N \simeq A^N \to e_1(A) \boxtimes \cdots \boxtimes e_N(A) = \mathrm{Ess}(A)$.

Concrete Generators

1. Translation to A: $e_{i_0}(A) \hookrightarrow A \simeq \tilde{A} \subseteq \mathrm{Ess}(A)^M$.
   for $j \in \{1, \ldots, M\}, i_0 \in \{1, \ldots, N\}$
   $$(x_i)_{i=1}^N \mapsto (e_i g_{ij}(x_{i_0}))_{i=1}^N.$$
2. Copy of generators: $\tilde{f}_s : A^{a_s} \longrightarrow \tilde{A}$.
   for $s \in \{1, \ldots, n\}$ and $j_0 \in \{1, \ldots, M\}$
   $$((x_{ijk})_{i=1}^N)_{1 \le j \le M, 1 \le k \le a_s} \mapsto (e_{i'} g_{i' j_0}(f_s(t(x_{ijk})_{i,j})_k))_{i'=1}^N.$$
   where a_s is the arity of f_s.
3. Recover from A: $\tilde{A}^N \simeq A^N \longrightarrow e_1(A) \boxtimes \cdots \boxtimes e_N(A) = \mathrm{Ess}(A)$.
   $$((x_{iji'})_{i=1}^N)_{j,i'} \mapsto (e_{i'}(t(x_{iji'})_{i,j})_{i'=1}^N.$$

"Embedding" $A \longrightarrow \mathrm{Ess}(A)^M$: $x \mapsto (e_i g_{ij}(x))$.
"retraction" $\mathrm{Ess}(A)^M \longrightarrow A$: $(x_{ij})_{1 \le i \le N, 1 \le j \le M} \mapsto t(x_{ij})_{ij}$

1. Keith A. Kearnes, Tame Congruence Theory is a localization theory, Lecture Notes from "A Course in Tame Congruence Theory" Workshop, Budapest, 2001.
2. Mike Behrisch, Relational Tame Congruence Theory and subalgebra primal algebras. Master's thesis, TU Dresden, 2009.
3. C. Bergman and J.Berman. Algorithms for categorical equivalence, Mathematical Structures in Computer Science vol 8 (1998) 1-15.
4. Ralph McKenzie. An algebraic version of categorical equivalence for varieties and more general algebraic categories, Logic and Algebra vol.180(1996)
5. K. Denecke, O. Lüders, Categorical equivalence of varieties and invariant relations, Algebra Universalis 46(2001) 105-118.
6. Shohei Izawa, Composition of matrix products and categorical equivalence, Algebra universalis 69(2013) 327-356.