Low growth equational complexity

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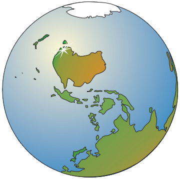

Equational complexity

$\mathbf{A}, \mathbf{B}$ finite algebras, same signature

$$\beta_{\mathbf{A}}(\mathbf{B}) = \min\{1, |p \approx q| : \mathbf{A} \models p \approx q \text{ and } \mathbf{B} \not\models p \approx q\}$$

$$\beta_{\mathbf{A}}(n) := \max\{\beta_{\mathbf{A}}(\mathbf{B}) : |B| < n\}.$$

The equational complexity of **A** is the function measuring the worst case shortest equation witnessing non membership of finite algebras in $\mathbb{HSP}(\mathbf{A})$.

If **A** has a finite equational basis, then $\beta_{\mathbf{A}}$ is eventually constant.

A finite algebra **A** is a counterexample to the *Eilenberg-Schützenberger conjecture* (1976) if and only if it is both without a finite equational basis but has $\beta_{\mathbf{A}}$ constantly bounded.

There are no known counterexamples to the E-S conjecture.

Typical growths

“Typical” growth.

- “Typical” growth rates $\beta_{\mathbf{A}}(n) \in \Theta(n^c)$ for constant c.
- If $\beta_{\mathbf{A}}(n) \in \Theta(n^c)$, then variety membership is in `pspace` (hence `exptime`). In fact in $\Pi_2^{\mathbb{P}}$. (McNulty, Szekely, Willard 2008; J and McNulty 2011)

Bad growth

Exponential growth rates are possible (Kozik 2007). Doubly exponential growth is an upper bound.

For unbounded $\beta_{\mathbf{A}}$ is it possible that $\beta_{\mathbf{A}}(n) \in O(\log(n))$? Or $O(\log(\log(n)))$? Or lower?

Quasi-equational complexity

Quasi-equational complexity

Defined in the same way, but using quasiequations: the shortest length of quasiequations required to guarantee exclusion of $< n$-element algebras from the quasivariety of **A**.

Observation

- $\mathbf{B} \notin \mathbb{SP}(\mathbf{A})$, if and only if there are $a \neq b$ in B such that every homomorphism from **B** to **A** identifies a and b.
- This is equivalent to satisfaction by **A** of the quasiequation

$$\text{diag}(\mathbf{B}) \to a \approx b$$

where the LHS is the positive atomic diagram of **B**.

Some group examples

Example.

The group C_3 versus the group C_2 (on $\{e, a\}$). Every $\phi : C_2 \to C_3$ has $\phi(e) = \phi(a)$. So

$$\mathbb{Z}_2 \models \begin{array}{c|cc} \cdot & e & a \\ \hline e & e & a \\ a & a & e \end{array} \to e \approx a$$

Equivalent to $aa = e \to a = e$.

(Technicality: “presentation” is defined relative to some variety.)

Short presentation conjecture

Babai, Goodman, Kantor, Luk and Pálfy (2000)

Conjecture: every finite group G has a presentation has a presentation of size $O(\log^3(n))$.

This conjecture is verified for very broad classes: solvable groups for example.

Ol'shanskiĭ (1974)

A finite group has a finite basis for its quasi-equations if and only if its Sylow subgroups are abelian.

Using these, one can find groups with unbounded quasi-equational complexity in $O(\log^3(n))$.

$\beta_{\mathbf{A}}$-critical algebras

A finite algebra **B** is $\beta_{\mathbf{A}}$-critical if $\beta_{\mathbf{A}}(\mathbf{B}) = \beta_{\mathbf{A}}(n)$ and $\beta_{\mathbf{A}}(n) > \beta_{\mathbf{A}}(n-1)$.

- If **B** is $\beta_{\mathbf{A}}$-critical, then it is subdirectly irreducible.

Theorem (corollary of J 2008).

Fix a finite group G and consider the flat extension $\flat(\mathbf{G})$. There is a finite system of equations Σ satisfied by $\flat(\mathbf{G})$ such that any subdirectly irreducible algebra satisfying Σ is isomorphic to a flat extension of a group in the variety generated by **G**.

And a flat algebra is in $\mathbb{HSP}(\flat(\mathbf{G}))$ if and only if it is isomorphic to $\flat(\mathbf{H})$ where **H** is a group in $\mathbb{SP}(\mathbf{G})$.

Theorem

For any finite nilpotent group G of nilpotency class 2, the equational complexity of the algebra $\flat(\mathbf{G})$ is in $O(\log^3(n))$.

Variety membership

Membership of finite algebras in the variety $\flat(\mathbf{G})$ is always polynomial time. In fact membership in the quasivariety of any finite group is always solvable in polynomial time (McGuire 2010).

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