Dualisability of relational structures

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Outline

- Generalising the background theory
- Natural duality for the class of n-colourable graphs
- Dualisability of bipartite graphs
- Dualisability of 3-colourable graphs
- Alternating chains of 3-colourable graphs
- Comparing the dualisability problems for algebras and relational structures

Generalising the background theory

A relational structure $\mathbf{M} = \langle M; R \rangle$ consists of an underlying set M and a set R of finitary relations on M.

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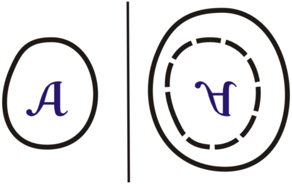
Most of the general theory of natural dualities for algebras still works for relational structures. [Hofmann, Davey]

Dualities

Duality

$\mathbf{M} = \langle M; R \rangle$

$\mathcal{A} = \mathbb{ISP}(\mathbf{M})$



$\underset{\sim}{\mathbf{M}} = \langle M; G, H, S, \mathcal{T} \rangle$

$\mathcal{X} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{M}})$

n-colourable graphs

Theorem

Let $n \in \mathbb{N}$. There exists a finite structure Γ_n such that $\mathbb{ISP}(\Gamma_n)$ is the class of all n-colourable graphs. [Wheeler, 1979]

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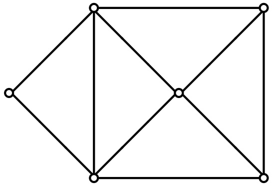

Γ_3

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Theorem

Let $n \in \mathbb{N}$. There exists a finite structure Γ_n such that $\mathbb{ISP}(\Gamma_n)$ is the class of all *n*-colourable graphs. [Wheeler, 1979]

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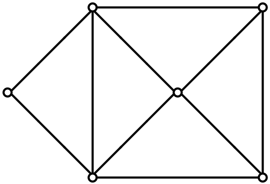

Γ_3

Theorem

For all $n \in \mathbb{N}$, there is a natural duality for the class consisting of all *n*-colourable graphs.

Freely n-colourable graphs

Definition

For $n \in \mathbb{N}$, let $\mathbf{K}_n$ be the complete simple graph on the set $\{1, \ldots, n\}$ and, for $n \geqslant 2$, let $\mathbf{K}_n^*$ be the graph $\mathbf{K}_n$ with one edge removed.

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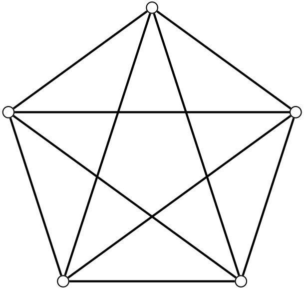

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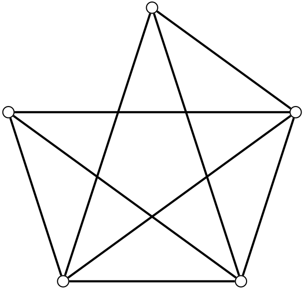

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Definition

A simple graph $\mathbf{A} = \langle A; r \rangle$ is *freely n-colourable* if it is n-colourable and, for every $u, v \in A$ with $u \neq v$ and $(u, v) \notin r$, there exist homomorphisms $\varphi_1, \varphi_2 \colon \mathbf{A} \to \mathbf{K}_n$ such that $\varphi_1(u) = \varphi_1(v)$ and $\varphi_2(u) \neq \varphi_2(v)$.

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Lemma

For each $n \in \mathbb{N}$, the class $\mathbb{ISP}(\mathbf{K}_n)$ consists of all freely n-colourable graphs.

Freely n-colourable graphs

Lemma

Let $\mathbf{M} = \langle M; r \rangle$ be a finite simple graph and let $n \in \mathbb{N}$ be such that $\mathbf{K}_n$ embeds into $\mathbf{M}$ but $\mathbf{K}_{n+1}$ does not embed into $\mathbf{M}$. If $n \geqslant 3$ and $\mathbf{K}^*_{n+1}$ does not embed into $\mathbf{M}$, then $\mathbf{M}$ is non-dualisable.

Freely n-colourable graphs

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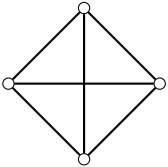

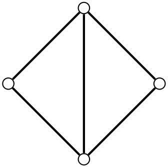

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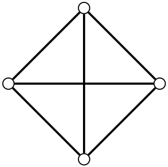

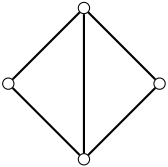

Non-dualisable

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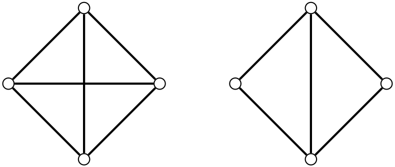

Non-dualisable

Corollary

If $n \geqslant 3$, then $\mathbf{K}_n$ is non-dualisable

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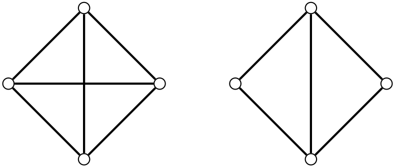


Non-dualisable

Corollary

If $n \geqslant 3$, then $\mathbf{K}_n$ is non-dualisable (so the class $\mathbb{ISP}(\mathbf{K}_n)$ of freely n-colourable graphs has no natural duality).

Bipartite Graphs

Bipartite Graphs

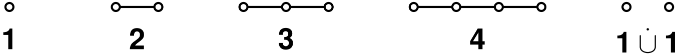
$\mathbf{1}$ $\mathbf{2}$ $\mathbf{3}$ $\mathbf{4}$ $\mathbf{1} \,\dot{\cup}\, \mathbf{1}$

Let **M** be a finite bipartite graph. Then

$$\mathbb{ISP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M}'), \text{ for some } \mathbf{M}' \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1} \,\dot{\cup}\, \mathbf{1}\}.$$

[Caicedo 1995]

Bipartite Graphs



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[Caicedo 1995]

Theorem

Every finite bipartite graph is dualisable.

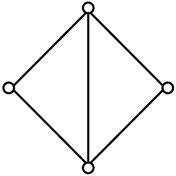
3-colourable graphs

3-colourable graphs

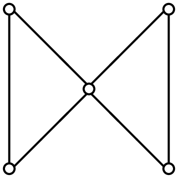
Theorem

Let **M** be a finite 3-colourable graph containing a cycle of length 3. Then **M** is dualisable if and only if $\mathbf{K}_4^*$ embeds into **M**.

$\mathbf{K}_4^*$



Example



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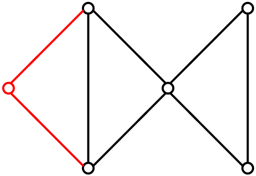

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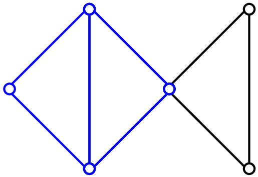

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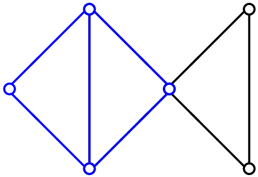

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Lemma

The graph $\mathbf{K}_4^*$ is inherently dualisable within the class of 3-colourable graphs.

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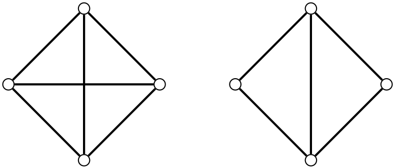

Non-dualisable

Alternating chains

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Theorem

Assume that $n \in \mathbb{N}$ with $2 \leqslant n$. There exist 3-colourable graphs $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ such that $\mathbf{A}_1 \leqslant \mathbf{A}_2 \leqslant \cdots \leqslant \mathbf{A}_n$ and the graphs are alternately dualisable and non-dualisable.

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$n = 6$

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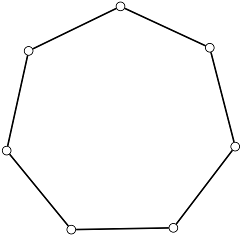
$\mathbf{A}_1$ Non-dualisable



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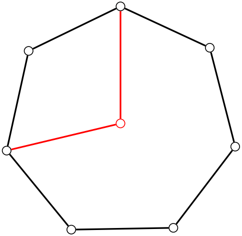

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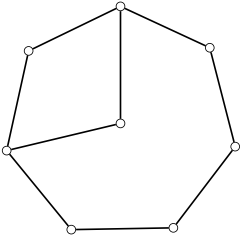
$\mathbf{A}_2$ Dualisable



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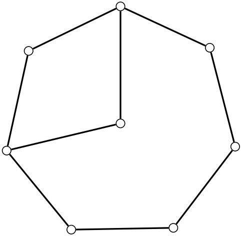
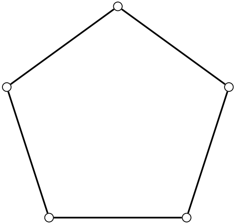
$\mathbf{A}_3$ Non-dualisable





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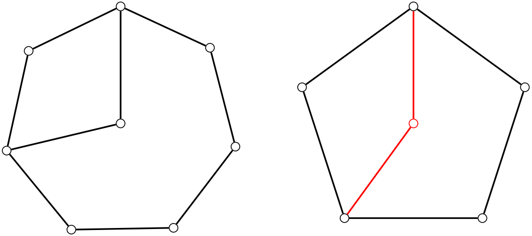

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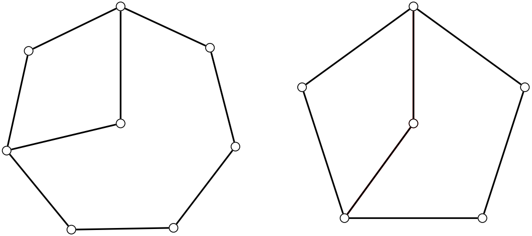
$n = 6$

$\mathbf{A}_4$ Dualisable



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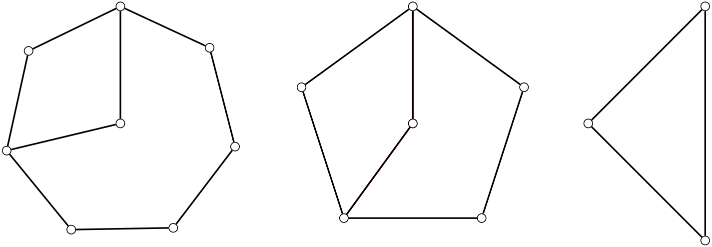
$\mathbf{A}_5$ Non-dualisable



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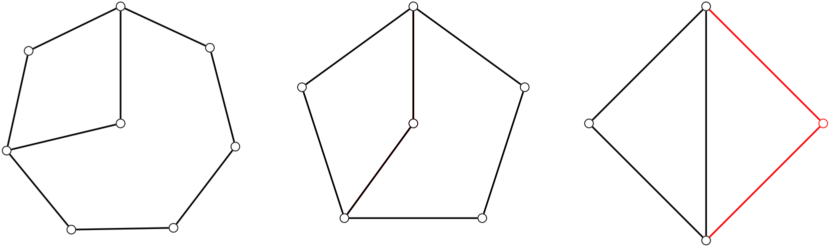

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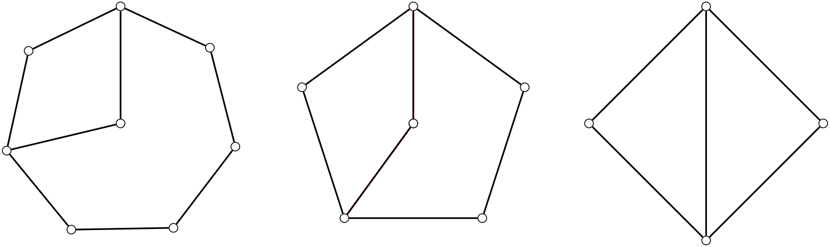
$n = 6$

$\mathbf{A}_6$ Dualisable



Graphs of algebras

Definition

Let $\mathbf{M} = \langle M; F \rangle$ be a finite algebra. We define the graph of $\mathbf{M}$ to be the relational structure

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Lemma

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Lemma

Let $\mathbf{M} = \langle M; F \rangle$ be a finite algebra. If $G(\mathbf{M})$ is dualisable, then $\mathbf{M}$ is also dualisable.

Example

Let $\mathbf{M} := \langle \{0, 1\}; \rightarrow \rangle$. Then $G(\mathbf{M}) = \langle \{0, 1\}; \text{graph}(\rightarrow) \rangle$ is non-dualisable.

Graphs of unars

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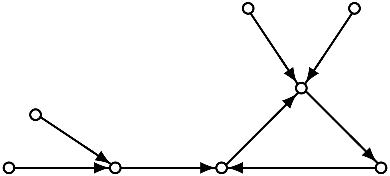

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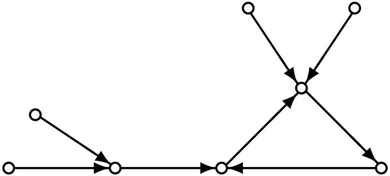

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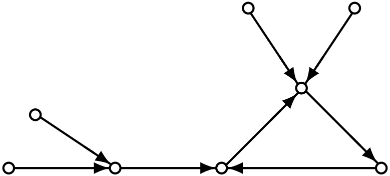

Theorem

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Corollary

Every finite unar is dualisable [Pitkethly].

Dualisability problems - Differences

$\langle \{0, 1\}; \rightarrow \rangle$ is non-dualisable.

Dualisability problems - Differences

$\langle\{0,1\};\rightarrow\rangle$ is non-dualisable. $\langle\{0,1\};\rightarrow,0\rangle$ is dualisable.

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$\langle\{0,1\};\rightarrow\rangle$ is non-dualisable. $\langle\{0,1\};\rightarrow,0\rangle$ is dualisable.

Algebras

- Can add operations to a non-dualisable algebra to get a dualisable algebra.

Relational structures

- Cannot add relations to a non-dualisable relational structure to get a dualisable relational structure.

Dualisability problems - Differences

Definition

M is said to be inherently dualisable if every finite structure into which **M** embeds is dualisable.

Similarly, **M** is inherently non-dualisable if every finite structure into which **M** embeds is non-dualisable.

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- ‘Many’ inherently dualisable relational structures.
- No inherently non-dualisable relational structures.

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