On the effect of natural dualities on questions regarding essential variables

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gaia, Melbourne, 2013

Essential variables.

$f: A^n \to B$ function.

Definition

The i-th variable essential of f is nonessential

$$:\iff f(x_1, \ldots, x_n) \approx f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).$$

The essential arity of f is the number of its essential variables.

What is already known...

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A lot

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A lot

so let's try a somewhat different approach...

Let's look at this with category theory...

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Let's look at this with category theory...

$f\colon A^n \to B$ function.

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$\Leftrightarrow f \circ \langle \pi_1^{n+1}, \dots, \pi_n^{n+1} \rangle = f \circ \langle \pi_1^{n+1}, \dots, \pi_{i-1}^{n+1}, \pi_{n+1}^{n+1}, \pi_{i+1}^{n+1}, \dots, \pi_n^{n+1} \rangle$

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~~$f: A^n \to B$ function~~ $f: \mathbf{A}^n \to \mathbf{B}$ morphism in category $\mathcal{C}$.

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$$\mathbf{A}^{(-)}\colon \quad \begin{array}{rcl} \mathbb{N} & \longrightarrow & \mathcal{C} \\ \mathbf{n} & \longmapsto & \mathbf{A}^n \\ \varphi\colon \mathbf{k} \to \mathbf{n} & \longmapsto & \langle \pi_{\varphi(1)}^n, \ldots, \pi_{\varphi(k)}^n \rangle \colon \mathbf{A}^n \to \mathbf{A}^k. \end{array}$$

$$\mathbf{n} := \{1, \ldots, n\}$$

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$$\Leftrightarrow f \circ \mathbf{A}^{\subseteq_n^{n+1}} = f \circ \mathbf{A}^{\psi_i^n}$$

where $\psi_i^n\colon \mathbf{n} \to \mathbf{n+1}\colon j \mapsto \begin{cases} j & \text{if } j \neq i, \\ n+1 & \text{if } j = i. \end{cases}$

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Dualizing this.

$f\colon \mathbf{A}^n \to \mathbf{B}$ operation.

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$g\colon \mathbf{Y} \to n \cdot \mathbf{X}$ dual operation.

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Dualizing this.

$g\colon \mathbf{Y} \to n \cdot \mathbf{X}$ dual operation.

The i-th variable of g is nonessential

$$:\Longleftrightarrow \quad \subseteq_n^{n+1} \cdot \mathbf{X} \circ g = \psi_i^n \cdot \mathbf{X} \circ g$$

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General abstract nonsense?

What's the difference after dualization?

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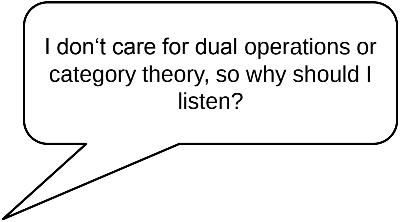
In abstract categories:	Symbols were pushed.
In concrete categories:	Meaning changed.

Every slide needs a title

I don‘t care for dual operations or category theory, so why should I listen?

Every slide needs a title



We'll get there...

The change of perspective.

$\mathcal{C}$ concrete category, $g\colon \mathbf{Y} \to n \cdot \mathbf{X}$.

Lemma

For each $\varphi\colon \mathbf{k} \to \mathbf{n}$, tfae:

(a) $g[\mathbf{Y}] \subseteq \varphi \cdot \mathbf{X}[k \cdot \mathbf{X}]$.

(b) *For each $i \in \{1, \ldots, n\} \setminus \varphi[\mathbf{k}]$, the i-th variable of g is nonessential.*

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How useful and easy is it to apply this lemma?

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Question

How useful and easy is it to apply this lemma?

Answer

That depends on the copowers of $\mathbf{X}$.

Very easy copowers.

$\mathcal{C} = \mathcal{S}et$

Proposition

Let $g \colon Y \to n \cdot X$.

$$i\text{-th variable nonessential} \iff g[Y] \cap \iota_i^n[X] = \emptyset.$$

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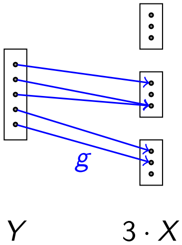

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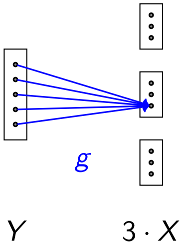


Essentially unary

Less easy copowers.

$\mathcal{C} = \mathcal{BoolGrp}$

- $n \cdot \mathbf{X} = \mathbf{X}^n$
- $\iota_i^n \colon \mathbf{X} \to n \cdot \mathbf{X},\ x \mapsto (0, \ldots, 0, \underset{\underset{i}{\uparrow}}{x}, 0, \ldots, 0)$
- $[g_1, \ldots, g_n] \colon n \cdot \mathbf{X} \to \mathbf{Y},\ (x_1, \ldots, x_n) \mapsto g_1(x_1) + \ldots + g_n(x_n)$

Proposition

Let $g \colon \mathbf{Y} \to n \cdot \mathbf{X}$.

i-th variable nonessential $\iff (\pi_i^n \circ g)(y) = 0$ for all $y \in Y$.

Horrible copowers.

they exist.

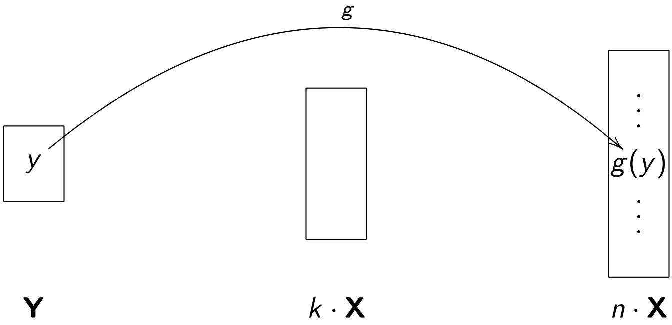
Some dual operations are always easy

Definition

An an n-ary dual operation $g\colon \mathbf{Y} \to n \cdot \mathbf{X}$ is said to respect the images of the injection morphisms to the degree k provided that

$$g[\mathbf{Y}] \subseteq \bigcup_{\varphi\colon \mathbf{k} \to \mathbf{n}} \varphi \cdot \mathbf{X}[k \cdot \mathbf{X}].$$

Below this line there is a picture.



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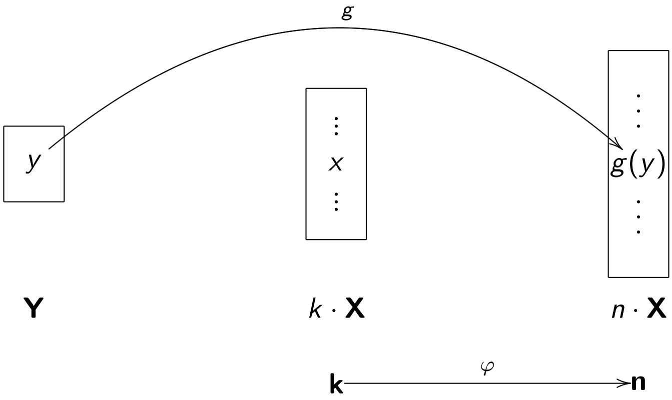

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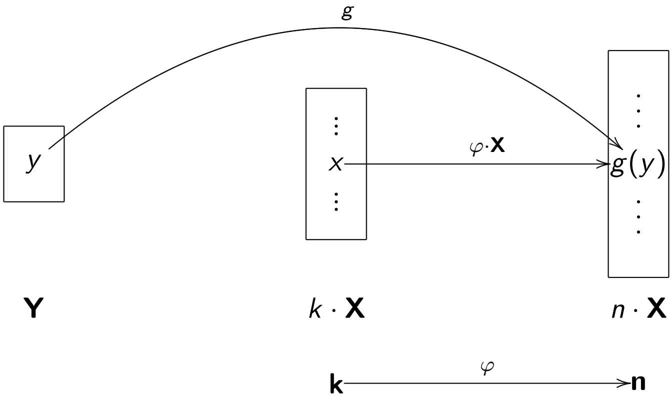

This is a natural property.

In many everyday-categories, all dual operations respect the images of the injection morphisms to the degree 1.

$\mathcal{Set}$, $\mathcal{Top}$, $\mathcal{Graph}$, $\mathcal{Priestley}$, $\mathcal{Stone}$, $\mathcal{Isbell}$, $\mathbb{ISP}(\mathcal{R})$...

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In many everyday-categories, all dual operations respect the images of the injection morphisms to the degree 1.

$\mathcal{Set}, \mathcal{Top}, \mathcal{Graph}, \mathcal{Priestley}, \mathcal{Stone}, \mathcal{Isbell}, \mathbb{ISP}(\mathcal{R})$...

Important for us:

It makes investigating essential variables much easier.

Tada.

$g\colon \mathbf{Y} \to n \cdot \mathbf{X}$ respects the images of the injection morphisms to the degree k.

Lemma

For $i \in \{1, \dots, n\}$, tfae:

(a) $g[\mathbf{Y}] \subseteq \bigcup_{\substack{\varphi\colon \mathbf{k} \to \mathbf{n} \\ i \notin \varphi[\mathbf{k}]}} \varphi \cdot \mathbf{X}[k \cdot \mathbf{X}]$,

(b) *g does not depend on its i-th variable.*

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Lemma

For $t \in \{1, \ldots, n\}$, tfae:

(a) $g[\mathbf{Y}] \subseteq \bigcup_{\substack{j \in \{1,\ldots,n\} \\ i \neq j}} \iota_j^n[\mathbf{X}]$,

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Proposition

The arity gap of g is 1 (and a lot of other stuff holds, too).

In fact...

Let $\mathbf{Y}$ be finite.

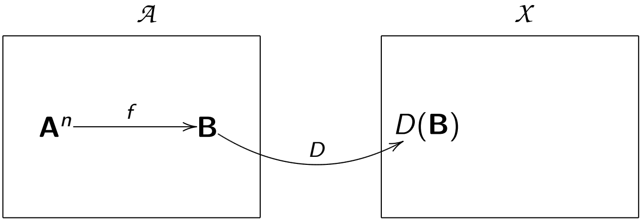
Theorem

Let $G \subseteq \{g\colon \mathbf{Y} \to n \cdot \mathbf{X} \mid n \in \mathbb{N}\}$.

Essential arity of dual operations among G is bounded	$\iff$	*$\exists k \in \mathbb{N}$: Each $g \in G$ respects the images of the injection morphisms to the degree k.*

Wake up you don't care for dual operations: This also helps for operations!

Let $\mathcal{A}$ and $\mathcal{X}$ be dually equivalent categories.



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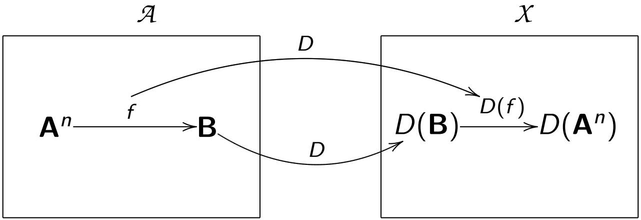

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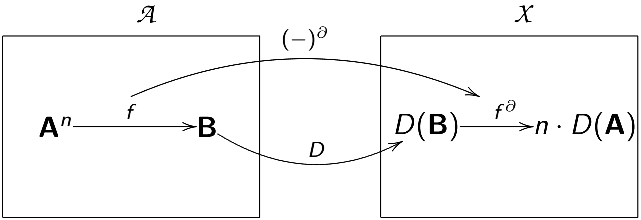

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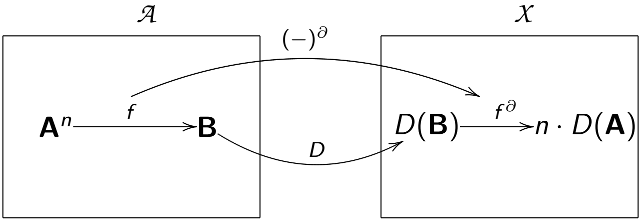

i-th variable of f essential $\iff$ i-th variable of f^∂ essential.

The easiest example I can think of.

Let $\mathbf{A}$, $\mathbf{B}$ be finite Boolean Algebras.

Question:

What is $|\{f : \mathbf{A}^n \to \mathbf{B} \mid f \text{ essential}\}|$?

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Using duality between $\mathcal{BoolAlg}_{fin}$ and $\mathcal{Set}_{fin}$:

$$|\{f : \mathbf{A}^n \to \mathbf{B} \mid f \text{ essential}\}|$$
$$= \{g \in \mathcal{Set}(\mathrm{At}(\mathbf{B}), n \cdot \mathrm{At}(\mathbf{A})) \mid g \text{ essential}\}|$$

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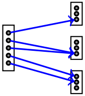

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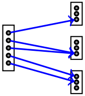

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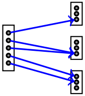

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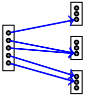

A slightly less easy example.

Let $\mathbf{A} = \langle A, m_{\mathbf{A}} \rangle$, $\mathbf{B} = \langle B, m_{\mathbf{B}} \rangle$ be finite median algebras.

Question:

What is a sharp bound on the essential arity of
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By dualizing F into the category of Isbell spaces:
Greatest number of prime ideals $I_1, \dots, I_k \subseteq A$ such that
$I_1, \dots, I_k, A \setminus I_1, \dots, A \setminus I_k$ is an antichain w.r.t. inclusion.

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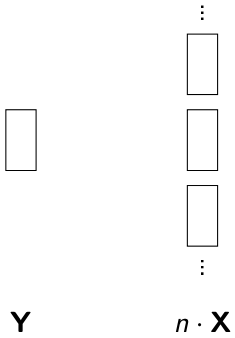
$$I \subseteq A \text{ Ideal} :\Longleftrightarrow \forall x \in A,\ y, z \in I : m_{\mathbf{A}}(x, y, z) \in I.$$

There are more examples like this.

There are similar results for structures with "nice" dualities (e.g, Boolean groups, distributive lattices, semilattices).

It also allows us to work with topology.

$\mathcal{C} = \mathcal{Top}$, $g : \mathbf{Y} \to n \cdot \mathbf{X}$



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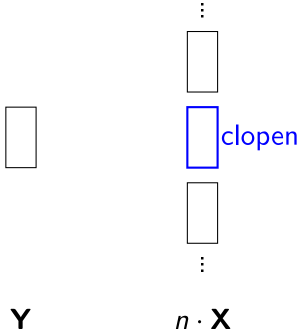

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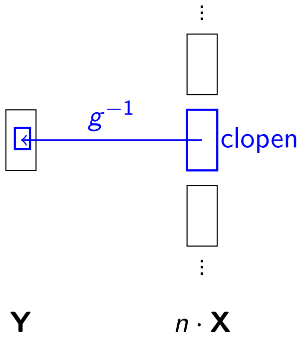

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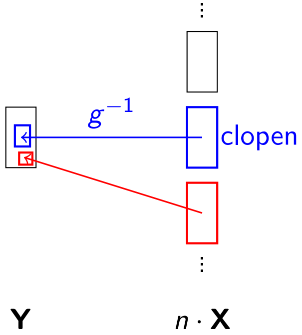

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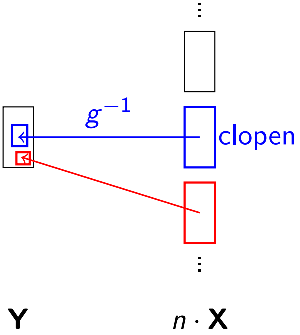

Essentiality of variables

$\longleftrightarrow$

Connectedness of $\mathbf{Y}$

Using this by taking the Gelfand Duality.

$F := \bigcup_{n\in\mathbb{N}} \mathrm{Hom}(\mathbf{A}^n, \mathbf{B})$.

Proposition

Let $\mathbf{A}, \mathbf{B}$ be comm. unital C^*-algebras. For each $k \in \mathbb{N}$, tfae:

a) $D(\mathbf{B})$ has exactly k connected components,

b) k is the supremum of the essential arity of operations in F,

c) the arity sequence of F is $(n!S(k,n)p_1)_{n\in\mathbb{N}}$ for $p_1 := |\mathbf{B}^{\mathbf{A}}|$.

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$F := \bigcup_{n\in\mathbb{N}} \mathrm{Hom}(\mathbf{A}^n, \mathbf{B})$.

Proposition

Let $\mathbf{A}, \mathbf{B}$ be comm. unital C^*-algebras. For each $k \in \mathbb{N}$, tfae:

a) $\mathbf{B}$ contains exactly 2^k idempotent elements,

b) k is the supremum of the essential arity of operations in F,

c) the arity sequence of F is $(n!S(k,n)p_1)_{n\in\mathbb{N}}$ for $p_1 := |\mathbf{B}^{\mathbf{A}}|$.

We can also use this to obtain general results...

A general result.

Theorem

Let $\mathcal{A}$ be a concrete category.
Let $\mathbf{B}$ be finite and

$$F \subseteq \{f \colon \mathbf{A}^n \to \mathbf{B} \mid n \in \mathbb{N}\},$$

the following statements are equivalent:

(a) *the essential arity of operations in F is bounded,*

(b) $\exists k \in \mathbb{N}$: *Each $g \in F^\partial$ respects the images of the injection morphisms to the degree k.*

A general result.

Theorem

Let $\mathcal{A}$ be a concrete category, dualized via a natural duality. Let $\mathbf{B}$ be finite, let the alter-ego $\mathbf{M}$ be a retract of $\mathbf{B}$ and

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the following statements are equivalent:

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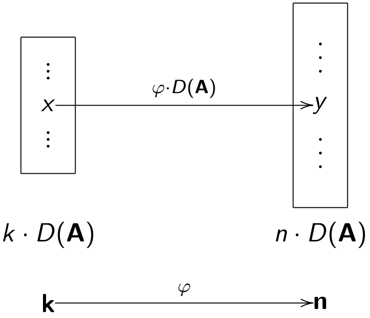
(b) $\exists k \in \mathbb{N} :$ *The copowers of $D(\mathbf{A})$ are non-deformed to the degree k (i.e., $\forall n \in \mathbb{N} : n \cdot D(\mathbf{A}) = \bigcup_{\varphi :\, \mathbf{k} \to \mathbf{n}} \varphi \cdot \mathbf{X}[k \cdot D(\mathbf{A})]$).*

Below this line there is again a picture.



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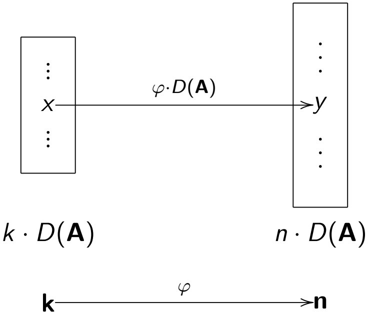

Remarks

- $k \neq 1$ is possible, but I don't know what it means...

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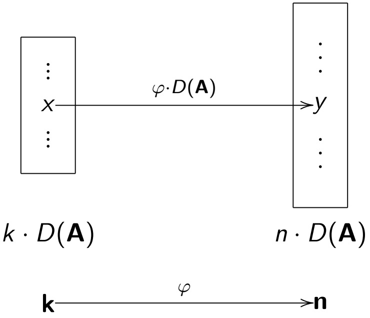

Remarks

- $k \neq 1$ is possible, but I don't know what it means...
- If $\mathbf{A}$ is finite and its powers are Cartesian, then it is enough to look at $n := \max(k + 2, |\mathbf{A}|)$.

Examples.

$$F := \bigcup_{n\in\mathbb{N}} \mathrm{Hom}(\mathbf{A}^n, \mathbf{B}),\ \mathbf{B} \text{ finite.}$$

$\mathbf{A}, \mathbf{B}$ are...	The essential arity is...
distributive lattices	bounded
Boolean groups	not bounded (unless $\lvert\mathbf{B}\rvert = 1$)

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$F := \bigcup_{n \in \mathbb{N}} \mathrm{Hom}(\mathbf{A}^n, \mathbf{B})$, $\mathbf{B}$ finite.

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Boolean groups	not bounded (unless $\lvert\mathbf{B}\rvert = 1$)

Interesting (well, I think so): The last statement does not hold in a dual version.

Conclusion.

- Dual operations offer a different (sometimes easier) view on essential arguments.

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