Algebra and CSP

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Definition (CSP)

Let $\mathbf{A}, \mathbf{B}$ be relational structures:

$$\mathbf{A} \xrightarrow{?} \mathbf{B}$$

Theorem (Geiger '68; Bodnarčuk, Kalužnin, Kotov, Romov '69)

There is a correspondence (given by Pol *and* Inv *operators) between relational structures* $\mathbf{B}$ *and algebras* $\mathbb{B}$*.*

In this talk:

- all the algebras are finite,
- all the algebras are idempotent (for every operation $t(x, \ldots, x) = x$)
- all of the relational structures have finite signature
- some of the relational structures are countably infinite.

Motivation and notation.

Width 1(Dalmau, Pearson '99)

1-local consistency checking algorithm for $\mathbf{A} \xrightarrow{?} \mathbf{B}$:

1. for every $a \in A$ set $B_a = B$
   - B_a is a set of "candidates" for homomorphic images of a;
2. pick $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ and $b_j \in B_{a_j}$ s.t.

$$R^{\mathbf{B}} \cap B_{a_1} \times \cdots \times \{b_j\} \times \cdots \times B_{a_n} = \emptyset$$

   and remove b_j from B_{a_j};
3. if possible repeat 2.

After the algorithm stops we have B_a's such that:

$$\text{if } (a_1, \ldots, a_n) \in R^{\mathbf{A}} \text{ then } R^{\mathbf{B}} \cap \prod_i B_{a_i} \subseteq_{sd} \prod_i B_{a_i}.$$

If B_a's are non-empty the algorithm succeeds but I will call it:

"establishing a homomorphism **UCT(A)** $\to$ **B**".

What is $\mathbf{UCT(A)}$

Fact (Kun, O'Donnell, Tamaki, Yoshida, Zhou '12)

For (countable) $\mathbf{A}$ *there is a canonical way to construct (countable)* $\mathbf{UCT(A)}$ *s.t.* $\mathbf{UCT(A)} \to \mathbf{B}$ *iff there are non-empty* B_a*'s such that:*

$$\text{if } (a_1, \ldots, a_n) \in R^{\mathbf{A}} \text{ then } R^{\mathbf{B}} \cap \prod_i B_{a_i} \leq_{sd} \prod_i B_{a_i}.$$

A $\mathbf{UCT(A)}$ comes with a homomorphism $\mathbf{UCT(A)} \xrightarrow{e} \mathbf{A}$:

(i) for every $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ exists $a'_1, \ldots, a'_n$ such that
- $e(a'_i) = a_i$ for all i and
- $(a'_1, \ldots, a'_n) \in R^{\mathbf{UCT(A)}}$.

(ii) there exists $\Sigma \leq \mathrm{Aut}(\mathbf{UCT(A)})$ such that:
- $e(v) = e(\sigma(v))$ for all v and $\sigma \in \Sigma$;
- for every v, v' such that $e(v) = e(v')$ there is $\sigma \in \Sigma$ s.t. $\sigma(v) = v'$.

If $\mathbf{UCT(A)} \xrightarrow{h} \mathbf{B}$ and $h(v) = h(v')$ whenever $e(v) = e(v')$ then $\mathbf{A} \to \mathbf{B}$.

$\mathbf{UCT}(\mathbf{A})$ is a tree

For $\mathbf{A} = (A, R_1, \ldots, R_n)$ we define a *multi-graph of* $\mathbf{A}$

- the vertex set is A,
- for every i and $(a_1, \ldots, a_l) \in R_i$ (for non-unary R_i's) we add edges $(a_1, a_2), (a_2, a_3), \ldots, (a_{l-1}, a_l)$.

A tree (Nešetřil, Tardif '00)

A relational structure is a tree if its multi-graph is a tree (i.e. no multiple edges, no cycles).

…and $\mathbf{UCT}(\mathbf{A})$ is a tree.

Width 1 (continued)

Assume that $\mathbb{B}$ – an algebra associated to $\mathbf{B}$ has $\wedge$-operation.

Question to verify is $\mathbf{A} \stackrel{?}{\to} \mathbf{B}$:

- imagine $\mathbf{UCT}(\mathbf{A}) \stackrel{e}{\to} \mathbf{A}$
- verify whether $\mathbf{UCT}(\mathbf{A}) \to \mathbf{B}$ (1-lcc)
  - if there is no homomorphism answer NO; else
  - define $H = \{f \in B^{UCT(\mathbf{A})} : \mathbf{UCT}(\mathbf{A}) \stackrel{f}{\to} \mathbf{B}\} \neq \emptyset$
  - $\mathbb{H} \leq \mathbb{B}^{UCT(\mathbf{A})}$
  - for $v, v' \in UCT(\mathbf{A})$ s.t. $e(v) = e(v')$
    - $\mathbb{B}_v = \mathbb{B}_{v'}$ (where $\mathbb{B}_w = \{h(w) : h \in H\}$) by (ii),
    - consider the element $\bigwedge \mathbb{H}$,
    - $\bigwedge \mathbb{H}(v) = \bigwedge \mathbb{H}(v')$;
  - conclude (using (i)) that the answer is YES.

Other consistency notions

The general idea of consistency checking algorithms for $\mathbf{A} \to \mathbf{B}$:

- imagine a relational structure $\mathbf{A}'$ (depends on the consistency we run (2, 3), Prague etc.) with $\mathbf{A}' \xrightarrow{e} \mathbf{A}$ and
- $\mathbf{A}'$ usually satisfies (i) and (ii).
- verify whether $\mathbf{A}' \to \mathbf{B}$,
- conclude that there is a solution to the original problem.

Absorption

Definition (Absorption)

Let $\mathbb{A}' \leq \mathbb{A}$. The algebra $\mathbb{A}'$ absorbs $\mathbb{A}$ ($\mathbb{A}' \lhd \mathbb{A}$) if there is a term t such that

$$t(A', \dots, A', A, A', \dots, A') \subseteq A \text{ (for any position of } A\text{)}.$$

Definition (Jónsson absorption)

Let $\mathbb{A}' \leq \mathbb{A}$. The algebra $\mathbb{A}'$ Jónsson absorbs $\mathbb{A}$ ($\mathbb{A}' \lhd_j \mathbb{A}$) if there is are terms $q_1, \dots, q_n$ such that $q_1(x, y, z) \approx x, q_n(x, y, z) \approx z$ and

$$q_i(x, y, y) \approx q_{i+1}(x, x, y) \text{ and } q_i(A', A, A') \subseteq A' \text{ for all } i.$$

- above is "directed version" the usual Jónsson identities on q_i's give equivalent definition;
- the same change can be made to Jónsson terms characterizing CD.

Bounded width

Assume that $\mathbb{B}$ – an algebra associated to $\mathbf{B}$ belongs to SD($\wedge$)-variety.

Question to verify is $\mathbf{A} \xrightarrow{?} \mathbf{B}$:

- imagine $\mathbf{PRG_B(A)} \xrightarrow{e} \mathbf{A}$ (it satisfies (i) and (ii))
- verify whether $\mathbf{PRG_B(A)} \to \mathbf{B}$ (some consistency)
  - if there is no homomorphism answer NO; else
  - define $H = \{f \in B^{PRG_{\mathbf{B}}(\mathbf{A})} : \mathbf{PRG_B(A)} \xrightarrow{f} \mathbf{B}\} \neq \emptyset$
  - $\mathbb{H} \leq \mathbb{B}^{PRG_{\mathbf{B}}(\mathbf{A})}$
  - for $v, v' \in PRG_{\mathbf{B}}(\mathbf{A})$ s.t. $e(v) = e(v')$
    - $\mathbb{B}_v = \mathbb{B}_{v'}$ (where $\mathbb{B}_w = \{h(w) : h \in H\}$) by (ii) – called potatoes;
  - if there is absorption we find $\mathbb{B}'_v \lhd \mathbb{B}_v$ such that
    - for $\mathbf{UCT(PRG_B(A))} \xrightarrow{e'} \mathbf{PRG_B(A)}$
    - there is homomorphism $\mathbf{UCT(PRG_B(A))} \xrightarrow{h} \mathbb{B}$ with $h(v) \in \mathbb{B}'_{e(e'(v))}$.
  - by some Lemma we get $\mathbf{PRG_B(A)} \xrightarrow{h'} \mathbf{B}$ such that $h'(w) \in \mathbb{B}_{e(w)}$.
  - and there is the non-absorbing case...
  - repeat and finitely conclude (using (i)) that the answer is YES.

New version of the lemma

Theorem

Let $\mathbf{A}$ *be countable relational structure and* $\mathbf{B}$ *be a finite one. Given*

$$\mathbb{B}'_a \lhd_j \mathbb{B}_a = \{h(a) : \mathbf{A} \xrightarrow{h} \mathbf{B}\}.$$

If, for some h'*,*

$$\mathbf{UCT}(\mathbf{A}) \xrightarrow{h'} \mathbf{B} \text{ and } \forall v \ h'(v) \in \mathbb{B}'_{e(v)} \qquad (\ddagger)$$

then, for some h*,*

$$\mathbf{A} \xrightarrow{h} \mathbf{B} \text{ and } \forall a \ h(a) \in \mathbb{B}'_a.$$

- given $\mathbf{A}, \mathbf{B}$ and $\mathbb{B}'_a$ if, for every $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$ we have

$$R^{\mathbf{B}} \cap \mathbb{B}'_{a_1} \times \cdots \times \mathbb{B}'_{a_n} \leq_{sd} \mathbb{B}'_{a_1} \times \cdots \times \mathbb{B}'_{a_n}$$

then $(\ddagger)$ holds.

Jónsson absorption + fin. related => absorption

Theorem (Barto, Bulin; K)

Let **B** *be a finite relational structure. If* $\mathbb{C} \triangleleft_j \mathbb{B}$ *then* $\mathbb{C} \triangleleft \mathbb{B}$.

Proof:

- take $\mathbf{B}^n$ for $\mathbf{A}$
- $\mathbb{H} \leq \mathbb{B}^{B^n}$ is the algebra of homomorphisms $\mathbf{B}^n$ to $\mathbf{B}$ ($\mathrm{Pol}_n(\mathbf{B})$)
- $\mathbb{B}_{(a_1,\dots,a_n)} = \{h(a_1,\dots,a_n) : h \in H\} = \mathrm{Sg}_{\mathbb{B}}(a_1,\dots,a_n)$
- putting $\mathbb{B}'_{(a_1,\dots,a_n)}$ to be:
  - $\mathbb{C} \cap \mathbb{B}_{(a_1,\dots,a_n)}$ if at most one a_i is outside $\mathbb{C}$;
  - $\mathbb{B}_{(a_1,\dots,a_n)}$ otherwise

  we get all except (‡)
- to get (‡) we use Barto's idea from CD=>NU (can be done for large enough n).

Nice transitive templates

Corollary

Let $\mathbf{B} = (B, R_1, \ldots, R_n)$ *be such that* $\mathrm{Aut}(\mathbf{B})$ *is transitive. Let* $\mathbb{B}$ *(idempotent!) has an absorbing subuniverse* $\mathbb{C}$ *such that every relation* $R_i \cap C^j \leq_{sd} C^j$. *Then* $\mathbf{B}$ *retracts into* $\mathbf{B}_{|C}$.

Proof:

- take $\mathbf{A}$ to be $\mathbf{B}$
- $\mathbb{H} \leq_{sd} \mathbb{A}^A$ is the algebra of endomorphisms
- set $\mathbb{B}'_a = \mathbb{C}$
- for every $(a_1, \ldots, a_j) \in R_i$ we have $R_i \cap C^j \subseteq_{sd} C^j$
- $\mathbb{H}$ contains an element in $\mathbb{C}^A$ – retraction follows.

Other transitive templates

Corollary

Let $\mathbf{B} = (B, R_1, \ldots, R_n)$ *be such that* $\mathrm{Aut}(\mathbf{B})$ *is transitive and* $\mathbb{B}$ *(idempotent!) is an algebra. For any* $\mathbf{A}$ *it is possible to find in a polynomial time* $\mathbb{B}'_a$ *such that:*

- $\mathbb{B}'_a$ *is a minimal absorbing subuniverse of* $\mathbb{B}$*, and*
- *if* $\mathbb{H} \neq \emptyset$ *then* $\mathbb{H} \cap \prod_a \mathbb{B}'_a \leq_{sd} \prod \mathbb{B}'_a$.

Proof:

- algebra of homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is empty or $\mathbb{B}_a = \mathbb{B}$;
- using consistency
  - either remove an element from some $\mathbb{B}_a$ and answer NO,
  - or find $\mathbb{B}'_a$ minimal absorbing and satisfying that for every $(a_1, \ldots, a_n) \in R^{\mathbf{A}}$

$$R^{\mathbf{B}} \cap \mathbb{B}'_{a_1} \times \cdots \times \mathbb{B}'_{a_n} \leq_{sd} \mathbb{B}'_{a_1} \times \cdots \times \mathbb{B}'_{a_n}$$

- so there is a homomorphism into $\mathbb{H} \cap \prod_a \mathbb{B}'_a$,
- subdirectness follows by minimality of $\mathbb{B}_a$'s.

Some intersections

Corollary

Let $\mathbb{C}_1, \ldots, \mathbb{C}_n \leq \mathbb{B}^k$ *such that* $\bigcap_j \mathbb{C}_j \leq_{sd} \mathbb{B}_k$*. Let* $\mathbb{B}'_i \lhd_j \mathbb{B}$ *and* $\mathbb{C}_j \cap \prod_i \mathbb{B}'_i \leq_{sd} \prod_i \mathbb{B}'_i$*; then* $\bigcap_j \mathbb{C}_j \cap \prod_i \mathbb{B}'_i \neq \emptyset$*.*

Proof:

- construct a relational structure $\mathbf{B} = (B, C_1, \ldots, C_n)$
- construct $\mathbf{A}$ on $\{1, \ldots, k\}$ by putting $(1, \ldots, k)$ into every $C_j^{\mathbf{A}}$
- the subdirectness condition is right there.

Algebraic length (Barto, K, Niven)

For a directed path the algebraic length of this path is the number of forward edges minus the number of backwards edges.

Corollary

Let $\mathbf{G} = (G, E)$ *be a directed, strongly connected digraph with a path from some vertex a to a of algebraic length one. If* $\mathbb{G}' \lhd \mathbb{G}$ *and* $E \cap (G')^2 \leq_{sd} (G')^2$*. Then there is path of algebraic length one from b to b in* $\mathbf{G}_{|G'}$*.*

Proof:

- join path's ends to obtain a graph **A**
- modify the graph to obtain new **A** such $\mathbb{H} \leq_{sd} \mathbb{G}^A$
- the subdirectness condition holds.

Someting about the proof.

Jónsson absorption with no inclusion

Definition (Jónsson absorption)

Let $\mathbb{A}', \mathbb{A}'' \leq \mathbb{A}$. The algebra $\mathbb{A}'$ Jónsson absorbs $\mathbb{A}$ ($\mathbb{A}' \nabla_J \mathbb{A}''$) if there is are terms $q_1, \ldots, q_n$ such that $q_1(x, y, z) \approx x$, $q_n(x, y, z) \approx z$ and

$$q_i(x, y, y) \approx q_{i+1}(x, x, y) \text{ and } q_i(A', A'', A') \subseteq A' \text{ for all } i.$$

Usually used for $\mathbb{A}', \mathbb{A}'' \leq \mathbb{B}^n$.

Definition

Let $\mathbf{B}$ be a relational structure. A relation $R \subseteq A^n$ is pp-definable in $\mathbf{B}$ if there exists a relation $\mathbf{A}$ such that $\pi_{1,\ldots,n} H = R$ (where H is a set of homomorphisms from $\mathbf{A}$ to $\mathbf{B}$).
The relation R is tree-pp-definable if $\mathbf{A}$ can be chosen to be a tree.

Main lemma

Lemma

Let $\mathbb{A}$ *be an algebra and* $\mathbb{C} = \{(a, \ldots, a) : a \in A\} \leq_{sd} \mathbb{A}^n$. *Let* $\mathbb{B} \leq_{sd} \mathbb{A}^n$ *and* $\mathbb{B} \nabla_j \mathbb{C}$. *Then* $\mathbb{B} \cap \mathbb{C} \neq \emptyset$.

Proof:

- it suffices to find $\mathbb{A}'$ such that $\mathbb{B} \cap (\mathbb{A}')^n \leq_{sd} (\mathbb{A}')^n$
- for $\emptyset \neq D, D' \subsetneq A$ define $D \sqsubseteq D'$ if D' can be tree-pp-defined using B and D.
- find maximal equivalence class (in $\sqsubseteq \cap \sqsupseteq$) above some $\{a\}$ (in $\sqsubseteq$) ;
- choose minimal under inclusion element of this class to be A';
- prove that A' is smallest (under inclusion) in this class;
- the end.