On Boolean-like algebras

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Joint thinking with T. Kowalski, F. Paoli, A. Salibra

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Motivation

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- What does it mean, for an algebra, to have a well-behaved Boolean core?
- What properties of Boolean algebras are responsible for the exceptionally rich and smooth structure theory of this variety?

Central elements

The concept around which our approach revolves, due to Vaggione, is that of a central element in a double pointed algebra, meaning an element which induces therein, in a specified sense, a pair of complementary factor congruences.

Given a similarity type ν including at least two (term-definable!) constants but otherwise arbitrary, we identify ν-algebras with a well-behaved Boolean core with ν-algebras having a retract of central elements, and Boolean ν-algebras with ν-algebras where every element is central.

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We can decompose the property of centrality into several equational properties, and we'll see what happens when some of them are satisfied but other ones are dropped.

This approach will give rise to a few successive approximations to a full-fledged notion of Boolean algebra of arbitrary similarity type.

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- $q(1, x, y) \approx x$;
- $q(0, x, y) \approx y$.

Church algebras

Definition

An algebra $\mathbf{A}$ of type ν is a *Church algebra* if there are term definable elements $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ s.t., for all $a, b \in A$,

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A variety $\mathcal{V}$ of type ν is a *Church variety* if every member of $\mathcal{V}$ is a Church algebra w.r.t. the same term $q\left(x, y, z\right)$ and the same constants $0, 1$.

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$\mathrm{Ce}(A)$ denotes the set of central elements of $\mathbf{A}$.

Examples

Variety	$q(x, y, z)$	*Central elements*
FL_{ew}-algebras	$(x \vee z) \wedge ((x \to 0) \vee y)$	Complemented elements
Orthomodular lattices	$(x \vee z) \wedge \left(x' \vee y\right)$	$e = (e \wedge a) \vee \left(e \wedge a'\right)$
Comm. rings with unit	$(x \oplus z)((1 - x) \oplus y)$	Idempotent elements
Combinatory algebras	$(x \cdot y) \cdot z$	

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$x \wedge y = q(x, y, 0)$ will be also denoted by $x \cdot y$ (or by plain juxtaposition) depending on the context.

Weak Boolean product representation

Theorem

Let $\mathbf{A}$ be a Church algebra. Then

$$\mathrm{Ce}\left(\mathbf{A}\right) = \left(\mathrm{Ce}\left(A\right); \vee, \wedge,', 0, 1\right)$$

is a Boolean algebra, isomorphic to the Boolean algebra of factor congruences of $\mathbf{A}$.

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This result, together with theorems by Comer and Vaggione, implies:

Theorem

Let $\mathbf{A}$ *be a Church algebra,* S *be the Boolean space of maximal ideals of* $\mathrm{Ce}(\mathbf{A})$ *and* $f : A \to \Pi_{I \in S} A/\theta_I$ *be the map defined by*

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where $\theta_I = \bigcup_{e \in I} \theta(0, e)$*. Then we have:*

1. f *gives a weak Boolean representation of* $\mathbf{A}$*.*
2. f *provides a Boolean representation of* $\mathbf{A}$ *iff, for all* $a \neq b \in A$*, there exists a least central element* e *such that*

   $$q(e, a, b) = a$$

   (i.e., $(a, b) \in \theta(0, e)$*).*

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$$A_e = \{e \wedge b : b \in A\}; \quad g_e(e \wedge \bar{b}) = e \wedge g(e \wedge \bar{b}).$$

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$$e \wedge g(\bar{b}) = e \wedge g(e \wedge \bar{b}),$$

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   It follows that

$$\mathbf{A} = \mathbf{A}_e \times \mathbf{A}_{e'}$$

Semi-central and central elements

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(Ax$_3$) $q(e, g(\overline{b}), g(\overline{c})) = g(q(e, b_1, c_1), \dots, q(e, b_n, c_n))$, for every $g \in \nu$.

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Therefore, central elements can be characterised equationally.

Semi-Boolean like algebras

Definition

A Church algebra $\mathbf{A}$ of type ν is a *semi-Boolean like algebra* (or a SBlA, for short) if every element of $\mathbf{A}$ is semi-central. I.e., for all $e, a, a_1, a_2, \overline{b}, \overline{c} \in A$:

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then $\mathbf{A}$ is a *Boolean like algebra* (or a BlA, for short).

Definition

A variety $\mathcal{V}$ of type ν is a (semi-)Boolean like variety if every member of $\mathcal{V}$ is a (semi-)Boolean like algebra with respect to the same term $q(x, y, z)$ and the same constants $0, 1$.

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- Orthomodular lattices, FL_{ew}-algebras or commutative rings with unit are not, in general, SBlAs, because of their failure to satisfy Ax$_1$
- Boolean algebras and Boolean rings are BlAs
- Any discriminator variety $\mathcal{V}$ with switching term s is a semi-Boolean like variety w.r.t. the term

$$q(e, x, y) = s(e, 0, y, x).$$

More examples

Examples of SBlAs which fail to be BlAs and fail to generate discriminator varieties:

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Example

Let $\mathbf{3} = (\{0, 1, 2\}; q, 0, 1)$be the Church algebra completely specified by the stipulation that $q(0, a, b) = q(2, a, b)$ for all $a, b \in \{0, 1, 2\}$. $\mathbf{3}$ *is semi-Boolean-like, but* $c(2) = q(2, 1, 0) = 0 \neq 2$. *Moreover,* $\mathbf{3}$ *is a nonsimple s.i. algebra, whence* $V(\mathbf{3})$ is not a discriminator variety.

Example

Let $\mathbf{3}' = (\{0, 1, 2\}\,; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(1, a, b) = q(2, a, b)$ *for all* $a, b \in \{0, 1, 2\}$. $\mathbf{3}'$ *is semi-Boolean-like, but* $c(2) = q(2, 1, 0) = 1 \neq 2$. *Moreover,* $\mathbf{3}'$ *is a nonsimple s.i. algebra, , whence* $V(\mathbf{3}')$ is not a discriminator variety.

In the case of semi-Boolean-like varieties, we could constructively specify the Church algebras representation theorem, In the case of semi-Boolean-like varieties, we could constructively specify the Church algebras representation theorem,

Theorem

Every SBlA $\mathbf{A}$ is a weak Boolean product of directly indecomposable semi-Boolean-like algebras.

Alternatively, the previous result can be seen as a consequence of the representation theorem for Church algebras: every SBlA has a weak Boolean product representation with directly indecomposable factors, because the directly indecomposable members of any SBlA variety can be defined via the universal formula

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$$\forall x \, (x \approx c\,(x) \Rightarrow x \approx 0 \veebar x \approx 1)\,.$$

A characterisation of semi-Boolean like varieties

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   - (ii) *for all $a \in \mathbf{A} \in \mathcal{V}$, $c(a)$ is central;*

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   - (i) *for all* $a, b, c \in \mathbf{A} \in \mathcal{V}$, $q(a, b, c) = q(c(a), b, c)$
   - (ii) *the universal formula* $c(0) \approx 0 \ \bar{\wedge} \ c(1) \approx 1 \ \bar{\wedge} \ \forall x(c(x) \approx 0 \veebar c(x) \approx 1)$ *holds in every directly indecomposable member of* $\mathcal{V}$.

The pure semi-Boolean like variety

The pure semi-Boolean like variety $\mathcal{SB}l\mathcal{A}_0$, consists of all the pure term reducts $(A; q, 0, 1)$ of semi-Boolean like algebras, and is axiomatised by Ax$_0$-Ax$_3$.

Theorem

Let $\mathbf{A}$ be a member of $\mathcal{SB}l\mathcal{A}_0$.

1. *The converse $\mathbf{A}^{\smile} = (A; q^{\smile}, 1, 0)$ of $\mathbf{A}$, where $q^{\smile}(x, y, z) = q(x, z, y)$, is also a member of $\mathcal{SB}l\mathcal{A}_0$ and the map $a \mapsto a'$ is an endomorphism from $\mathbf{A}$ into $\mathbf{A}^{\smile}$.*

Theorem

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1. *The converse* $\mathbf{A}^{\smile} = (A; q^{\smile}, 1, 0)$ *of* $\mathbf{A}$*, where* $q^{\smile}(x, y, z) = q(x, z, y)$*, is also a member of* $\mathcal{SB}l\mathcal{A}_0$ *and the map* $a \mapsto a'$ *is an endomorphism from* $\mathbf{A}$ *into* $\mathbf{A}^{\smile}$*.*
2. $c[\mathbf{A}]$ *is a retract of* $\mathbf{A}$*.*

Generators for the variety

Theorem

$V(\{\mathbf{3}, \mathbf{3}'\}) = \mathcal{SB}l\mathcal{A}_0$.

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Theorem

$\mathcal{SBlA}_0$ is generated as a variety by $\mathbf{4}$.

Congruence identities

Consider the class $\mathcal{K}$ of all finite algebras in $\mathcal{SBlA}_0$ which satisfy the condition

$$x \neq 1 \Rightarrow q\left(x, y, z\right) \approx z.$$

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is a congruence.
Every proper subpartition of θ corresponds to a congruence, which means that the lattice of congruences of $\mathbf{A}$ coincides with the full lattice of partitions of $A - \{1\}$.

Therefore:

Theorem

$\mathcal{SBlA}_0$ has no congruence identities.

As a consequence...

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- $\mathcal{SB}l\mathcal{A}_0$ is not congruence modular

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- $\mathcal{SB}l\mathcal{A}_0$ is not congruence modular
- $\mathcal{SB}l\mathcal{A}_0$ is not congruence n-permutable for any n

Moreover:

- $\mathcal{SB}l\mathcal{A}_0$ is neither 0- nor 1-subtractive

As a consequence...

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Moreover:

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However, it enjoys some (weaker) pleasant properties

- $\mathcal{SB}l\mathcal{A}_0$ is quasi-subtractive (indeed, quasi-discriminator) and weakly $\{c(x) \approx 1\}$-regular.

In every member $\mathbf{A}$ of $\mathcal{SB}l\mathcal{A}_0$ there is a lattice isomorphism between the lattice of $\mathcal{SB}l\mathcal{A}_0$-open filters of $\mathbf{A}$ and the lattice of congruences of $\mathbf{A}$ whose quotients belong to $\mathcal{B}l\mathcal{A}_0$.

Lemma

Let $\mathbf{A} \in \mathcal{SBl}\mathcal{A}_0$, and let $F \subseteq A$. Then t.f.a.e.:

1. *F is a $\mathcal{SBl}\mathcal{A}_0$-open filter of $\mathbf{A}$;*

Lemma

Let $\mathbf{A} \in \mathcal{SBlA}_0$, *and let* $F \subseteq A$. *Then t.f.a.e.:*

1. F *is a* $\mathcal{SBlA}_0$*-open filter of* $\mathbf{A}$;
2. F *satisfies the conditions* *(F1)* $1 \in F$; *(F2)* $a, b \in F \Rightarrow a \wedge b \in F$; *(F3)* $a \in F, b \in A \Rightarrow a \vee b, b \vee a \in F$; *(F4)* $c(x) \in F \Rightarrow x \in F$;

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3. F *satisfies F1, F4, and* *(G1)* $a, b \in F, d \in A \Rightarrow q(a, b, d), q(d, a, b) \in F$*.*

An algebraic characterisation (1)

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An analogue of this result is available in our context.

An algebraic characterisation (2)

Definition

Let $\mathcal{V}$ be a variety of type ν, and let t, s be at most unary terms of the same type. $\mathcal{V}$ is *(t, s)-permutable* iff for every $\mathbf{A} \in \mathcal{V}$, every $\theta, \psi \in \mathrm{Con}(\mathbf{A})$ and every $a, b \in A$, $(t(a), s(b)) \in \theta \circ \psi$ iff $(t(a), s(b)) \in \psi \circ \theta$.

An algebraic characterisation (2)

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Theorem

A variety $\mathcal{V}$ is (t, s)-permutable iff there exists a ternary term p such that

$$\mathcal{V} \vDash p(x, s(y), y) \approx t(x)$$
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An algebraic characterisation (3)

Let $\mathcal{V}$ be a Church variety. $\theta \in \mathrm{Con}\,(\mathbf{A})$ with $\mathbf{A} \in \mathcal{V}$, is *Boolean* iff $\mathbf{A}/\theta$ is a BlA.
We denote by $\theta_B\,(a, b)$ the smallest Boolean congruence collapsing a and b.

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We denote by $\theta_B(a,b)$ the smallest Boolean congruence collapsing a and b.

A Church variety $\mathcal{V}$ has *EDPBC* if there exists a finite system of quaternary terms $s_i(x,y,z,w), t_i(x,y,z,w), i < n$, such that, for every $\mathbf{A} \in \mathcal{V}$ and all $a,b,d,f \in A$,

$$d \equiv_{\theta_B(a,b)} f \text{ iff } \mathbf{A} \vDash s_i^{\mathbf{A}}(a,b,d,f) = t_i^{\mathbf{A}}(a,b,d,f), i < n.$$

A Church variety $\mathcal{V}$, where c is a $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible term, is Boolean semisimple iff in every s.i. member of $\mathcal{V}$ the only Boolean congruences are $\ker c$ and ∇.

An algebraic characterisation (4)

Theorem

For $\mathcal{V}$ a Church variety where c is a $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible term, t.f.a.e.:

1. *$\mathcal{V}$ is a semi-Boolean like variety;*

An algebraic characterisation (4)

Theorem

For $\mathcal{V}$ a Church variety where c is a $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible term, t.f.a.e.:

1. *$\mathcal{V}$ is a semi-Boolean like variety;*
2. *$\mathcal{V}$ is c-permutable, has EDPBC and is Boolean semisimple.*

Theorem

Let $\mathbf{A}$ be a BlA. The following conditions are equivalent:

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4. $|A| = 2$.

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Theorem

$\mathcal{B}l\mathcal{A}_0$ is term equivalent to $\mathcal{B}\mathcal{A}$.

Characterising Boolean like algebras

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1. $\mathbf{A}$ *is a* BlA*;*
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3. $\vee$ *is commutative;*
4. $\wedge$ *and* $\vee$ *are both idempotent.*

Idempotent semi-Boolean like algebras

Whereas idempotency of both join and meet is enough to enforce a Boolean like behaviour in a SBlA, idempotency of join alone (or meet alone) is not: the algebras $\mathbf{3}$ and $\mathbf{3}'$ are respective counterexamples.

Therefore, we may look for some middle ground between these concepts by adding either one of the idempotency identities.

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A SBlA is *(meet-)idempotent* if it satisfies the following identity:

(Ax_5) $x \wedge x \approx x$.

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Theorem

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(iv) $\mathcal{V}$ *is* 0*-subtractive with witness term* $x - y = y' \wedge x$*;*

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(iv) *$\mathcal{V}$ is 0-subtractive with witness term $x - y = y' \wedge x$;*

(v) *The identity*

$$x + 0 \approx x$$

holds in $\mathcal{V}$, where $x + y = q(x, x, y)$.

The pure idempotent semi-Boolean like variety $\mathcal{ISBlA}_0$, consisting of all the pure term reducts of id. SBlAs, is axiomatised by Ax_0-Ax_3 plus Ax_5.

Theorem

$V(\mathbf{3}') = \mathcal{ISBlA}_0$.

Noncommutative Boolean algebras

Noncommutative Boolean algebras play w.r.t. idempotent semi-Boolean like varieties the same role as left-handed skew Boolean $\cap$-algebras play w.r.t. pointed discriminator varieties.

Definition

A non-commutative Boolean algebra is an algebra $\mathbf{A} = (A; +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following conditions for all $a, b, c \in A$:

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(S0) $(A; +, \cdot)$ is a double band;

(S1) (i) $0 + a = a \qquad 1 + a = 1$;
(ii) $a + 0 = a$

(S2) $0a = a0 = 0; \qquad 1a = a$

(S3) (i) $a(b + c) = ab + ac$
(ii) $(b + c)a = ba + ca$

(S4) $(a + b)' = a'b'; \qquad (ab)' = a' + b'$

Noncommutative Boolean algebras (continued)

Definition

(S5) (i) $aa' = 0$
(ii) $a'a = 0$

(S6) $a + a' = a' + a = a + 1$

(S7) $ba + b'a = a$

(S8) $ca + c'b = c'b + ca$

(S9) $abc = bac$

An algebra $\mathbf{A} = (A; +, \cdot, ', 0, 1, g)_{g \in \nu}$ of type ν is said to be a noncommutative Boolean algebra with additional regular operations iff it is an NBA satisfying the identities

(S10) $g(\ldots, ex_i + e'y_i, \ldots) \approx e \cdot g(\ldots, x_i, \ldots) + e' \cdot g(\ldots, y_i, \ldots)$ $(g \in \nu)$.

A term equivalence result

Theorem

The varieties $\mathcal{ISBlA}_0$ and $\mathcal{NBA}$ are term equivalent via the correspondences

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$$\begin{array}{lcl} q(x,y,z) & \rightsquigarrow & xy + x'z \\ x + y & \rightsquigarrow & q(x,x,y) \\ xy & \rightsquigarrow & q(x,y,0) \\ x' & \rightsquigarrow & q(x,0,1) \end{array}$$

Theorem

Every idempotent semi-Boolean like variety is term equivalent to a variety of noncommutative Boolean algebras with additional regular operations.

Applications to discriminator varieties (1)

Theorem

Let $\mathcal{V}$ be a double pointed variety. Then $\mathcal{V}$ is discriminator if, and only if, $\mathcal{V}$ is 0-regular and idempotent semi-Boolean-like.

Corollary

A double pointed variety of type ν is a discriminator variety if, and only if, for suitable terms $q(x, y, z)$, $w(x, y, z)$ and $d(x, y)$, it satisfies the following identities:

- $x \approx q(1, x, y) \approx q(0, y, x) \approx q(y, x, x) \approx q(x, x, 0)$;
- $q(x, q(x, y_1, y_2), z) \approx q(x, y_1, z) \approx q(x, y_1, q(x, y_2, z))$;
- $q(x, g(\overline{y}), g(\overline{z}) \approx g(q(x, y_1, z_1), \ldots, q(x, y_n, z_n))$, *for every* $g \in \nu$;
- $d(x, x) \approx 0$;
- $x \approx w(x, y, 0) \approx w(y, x, d(y, x))$.

Applications to discriminator varieties (2)

Theorem

Let $\mathcal{V}$ be a double pointed variety. Then $\mathcal{V}$ is a discriminator variety if, and only if, $\mathcal{V}$ is an idempotent semi-Boolean-like variety and there exists a binary term $u(x, y)$ such that the identity

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Let $\mathcal{V}$ be a double pointed variety. Then $\mathcal{V}$ is a discriminator variety if, and only if, $\mathcal{V}$ is an idempotent semi-Boolean-like variety and there exists a binary term $u(x, y)$ such that the identity

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holds in $\mathcal{V}$, and

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Let $\mathcal{V}$ be a double pointed variety. Then $\mathcal{V}$ is a discriminator variety if, and only if, $\mathcal{V}$ is an idempotent semi-Boolean-like variety and there exists a binary term $u(x, y)$ such that the identity

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holds in $\mathcal{V}$, and the quasi-identity

$$x \neq y \Rightarrow u(x, y) \approx x$$

holds in every directly indecomposable member of $\mathcal{V}$.

Thank you!