

Maltsev conditions and directed graphs

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Definition

A **digraph** is a pair $\mathbb{G} = (G; \rightarrow)$, where G is the set of **vertices** and $\rightarrow \subseteq G^2$ is the set of **edges**.

Definition

A **homomorphism** from \mathbb{G} to \mathbb{H} is a map $f : G \rightarrow H$ that preserves edges:

$$a \rightarrow b \text{ in } \mathbb{G} \quad \Longrightarrow \quad f(a) \rightarrow f(b) \text{ in } \mathbb{H}.$$

$\text{Hom}(\mathbb{G}, \mathbb{H})$ is the set of all homomorphisms from \mathbb{G} to \mathbb{H} .

Definition

The clone of **polymorphisms** of \mathbb{G} is $\text{Hom}(\mathbb{G}) = \bigcup_{n=1}^{\infty} \text{Hom}(\mathbb{G}^n, \mathbb{G})$.

Theorem (Larose, Zádori; 1997)

If a finite poset (reflexive, transitive, antisymmetric digraph) has Gumm polymorphisms

$$\begin{aligned}x &\approx d_0(x, y, z), \\d_i(x, y, x) &\approx x \text{ for all } i, \\d_i(x, y, y) &\approx d_{i+1}(x, y, y) \text{ for even } i, \\d_i(x, x, y) &\approx d_{i+1}(x, x, y) \text{ for odd } i, \\d_n(x, y, y) &\approx p(x, y, y), \text{ and} \\p(x, x, y) &\approx y,\end{aligned}$$

then it has a near-unanimity polymorphism

$$n(y, x, \dots, x) \approx \dots \approx n(x, \dots, x, y) \approx x.$$

Theorem (Larose, Loten, Zádori; 2005)

If a finite reflexive and symmetric digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Theorem (M, Zádori; 2012)

If a finite reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism and totally symmetric polymorphisms

$$\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\} \implies t(x_1, \dots, x_n) \approx t(y_1, \dots, y_n)$$

for all arities.

Theorem

If a finite symmetric digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

HOW FAR CAN WE PUSH THIS?

Theorem (Kazda; 2011)

If a finite digraph has a Maltsev polymorphism

$$p(x, x, y) \approx p(y, x, x) \approx y,$$

then it admits a majority polymorphism

$$m(y, x, x) \approx m(x, y, x) \approx m(x, x, y) \approx x.$$

Theorem (Bulín, Delić, Jackson, Niven; 2013)

For every finite relational structure \mathbb{A} there exists a finite directed graph \mathbb{G} , such that almost all Maltsev conditions (Taylor term, Willard terms, Hobby-McKenzie terms, Gumm terms, edge term, Jonsson terms, near-unanimity term, but not Maltsev term) hold equivalently by \mathbb{A} and \mathbb{G} .

Definition

\mathbb{G} is **strongly connected** if for any $a, b \in G$ there exists a **directed path** $a = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n = b$ of length $n \geq 0$. \mathbb{G} is **connected** if for any $a, b \in G$ there exists an **oriented path** $a = a_0 \rightarrow a_1 \leftarrow \cdots \rightarrow a_n = b$ of length $n \geq 0$ where the arrows can point either way. The digraph \mathbb{G} is **smooth**, if its edge relation is subdirect (no sources and sinks).

Definition

The [strong, smooth] components of \mathbb{G} are the maximal [strong, smooth] induced subgraphs of \mathbb{G} .

Definition

The **algebraic length** of a directed path is the number of forward edges minus the number of backward edges. The algebraic length of \mathbb{G} is the smallest positive algebraic length of oriented cycles (closed paths) of \mathbb{G} .

ALGEBRAIC LENGTH 1

Proposition

If \mathbb{G}, \mathbb{H} are connected, smooth and \mathbb{G} has algebraic length 1, then $\mathbb{G} \times \mathbb{H}$ is connected and smooth.

Proof.

- for any $a \in G$ there exists an oriented cycle of algebraic length 1 going through a
- for any $a \rightarrow b$ in \mathbb{G} there exists an oriented path from a to b of algebraic length 0
- for any $x \in H$, (a, x) and (b, x) are connected in $\mathbb{G} \times \mathbb{H}$ □

Proposition

If \mathbb{G} is smooth, algebraic length 1 and (strongly) connected, then \mathbb{G}^n is smooth, algebraic length 1 and (strongly) connected for all $n \geq 1$.

Definition

Write $\mathbb{G} \rightarrow \mathbb{H}$ iff there exists a homomorphism from \mathbb{G} to \mathbb{H} .

Proposition

\rightarrow is a quasi-order on the set of finite digraphs. If \mathbb{G} is a minimal member of the \leftrightarrow class of \mathbb{H} , then

- every endomorphism of \mathbb{G} is an automorphism,
- \mathbb{G} is uniquely determined up to isomorphism, and
- \mathbb{G} is isomorphic to a induced substructure of \mathbb{H} .

Definition

\mathbb{G} is a **core** if it has no proper endomorphism. The **core of** \mathbb{H} is the uniquely determined core structure in the \leftrightarrow class of \mathbb{H} .

THE LOOP LEMMA

Theorem (Barto, Kozik; 2008)

If \mathbb{G} is smooth, algebraic length 1, and has a Taylor polymorphism, or equivalently a weak near-unanimity polymorphism

$$w(x, \dots, x) \approx x \quad \text{and} \quad w(y, x, \dots, x) \approx \dots \approx w(x, \dots, x, y),$$

then \mathbb{G} has a loop.

Corollary (Barto, Kozik, Niven; 2008)

The core of a smooth digraph with a Taylor polymorphism is a disjoint union of cycles.

Problem

Let \mathbb{G} be a smooth, connected, algebraic length 1 digraph that has Gumm polymorphisms. Does \mathbb{G} need to have a near-unanimity polymorphism?

Theorem

If $\mathbb{G} = (G; E)$ is smooth, connected, algebraic length 1, and has Maltsev polymorphism, then it has join and meet polymorphisms.

Proof.

- $\alpha = E \circ E^{-1}$ and $\beta = E^{-1} \circ E$ are equivalence relations (congruences)
- case 1: $(a, b) \in \alpha \wedge \beta$ and $a \neq b$
 - $r: \mathbb{G} \rightarrow \mathbb{G} \setminus \{b\}$, $r(x) = x$ for $x \neq b$ and $r(b) = a$ is a retraction
 - by induction we have join and meet polymorphisms on $r(\mathbb{G})$
 - we can extend them to \mathbb{G} by splitting $\{a, b\}$ into $a < b$
- case 2: $\alpha \wedge \beta = 0$
 - by induction the digraph $\mathbb{G}/\alpha = (G/\alpha; E/\alpha)$ has join and meet
 - the digraph $\mathbb{E}/\alpha = (E/\alpha; \pi_2 \circ \pi_1^{-1})$ has join and meet
 - the digraphs \mathbb{E}/α and \mathbb{G} are isomorphic via the map $\varphi: E/\alpha \rightarrow G$,
 $\varphi(x/\alpha, y/\alpha) = x/\beta \cap y/\alpha$



Definition

Let $\mathbb{H}^{\mathbb{G}}$ be the digraph on the set $H^{\mathbb{G}}$ with edge relation $f \rightarrow g$ iff

$$a \rightarrow b \text{ in } \mathbb{G} \implies f(a) \rightarrow g(b) \text{ in } \mathbb{H}.$$

Proposition

- $\text{Hom}(\mathbb{G}, \mathbb{H}) = \{f \in \mathbb{H}^{\mathbb{G}} : f \rightarrow f\}$
- $\mathbb{G}^n = \mathbb{G}^{\mathbb{L}_n}$ where $\mathbb{L}_n = (\{1, \dots, n\}; =)$
- $(\mathbb{H}^{\mathbb{G}})^{\mathbb{F}} = \mathbb{H}^{\mathbb{G} \times \mathbb{F}}$
- $\mathbb{H}^{\mathbb{F}} \times \mathbb{G}^{\mathbb{F}} = (\mathbb{H} \times \mathbb{G})^{\mathbb{F}}$
- the composition map $\circ : \mathbb{H}^{\mathbb{G}} \times \mathbb{G}^{\mathbb{F}} \rightarrow \mathbb{H}^{\mathbb{F}}$ is a homomorphism
- If $f \rightarrow g$ in $\mathbb{H}^{\mathbb{G}^n}$ and $f_1 \rightarrow g_1, \dots, f_n \rightarrow g_n$ in $\mathbb{G}^{\mathbb{F}}$, then

$$f(f_1, \dots, f_n) \rightarrow g(g_1, \dots, g_n) \text{ in } \mathbb{H}^{\mathbb{F}}$$

EXPONENTIATION IN FINITE DUALITY

- set of finite relational structures modulo \leftrightarrow is a partially ordered set
- isomorphic to the set of core isomorphism types
- minimal [maximal] element: 1-element structure, with empty [full] relations
- join: disjoint union, meet: direct product,
- satisfies distributive laws, join irreducible = connected
- Heyting algebra (relatively pseudocomplemented)
- $\mathbb{F} \wedge \mathbb{G} \leq \mathbb{H} \iff \mathbb{H}^{\mathbb{F} \times \mathbb{G}} = (\mathbb{H}^{\mathbb{G}})^{\mathbb{F}}$ has a loop $\iff \mathbb{F} \leq \mathbb{H}^{\mathbb{G}}$
- if \mathbb{G} is join irreducible with lower cover \mathbb{H} , then $(\mathbb{G}, \mathbb{H}^{\mathbb{G}})$ is a dual pair

Theorem (Nešetřil, Tardif, 2010)

Let \mathbb{G} be a finite connected core structure. Then \mathbb{G} has a dual pair \mathbb{H} , i.e. $\{\mathbb{F} \mid \mathbb{F} \rightarrow \mathbb{G}\} = \{\mathbb{F} \mid \mathbb{H} \not\rightarrow \mathbb{F}\}$, if and only if \mathbb{G} is a tree.

End(\mathbb{G}), Sym(\mathbb{G}) AND Aut(\mathbb{G})

Definition

End(\mathbb{G}) and Sym(\mathbb{G}) are the induced subgraphs of $\mathbb{G}^{\mathbb{G}}$ on $\text{Hom}(\mathbb{G}, \mathbb{G})$ and the set of permutations, respectively. $\text{Aut}(\mathbb{G}) = \text{End}(\mathbb{G}) \cap \text{Sym}(\mathbb{G})$.

Proposition

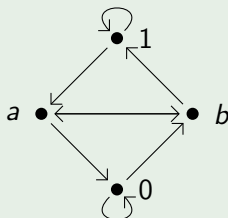
The components of Sym(\mathbb{G}) [End(\mathbb{G})] that contain an automorphism are isomorphic to the component of the identity.

Theorem (Gyenizse; 2013)

Aut(\mathbb{G}) is a disjoint union of complete digraphs. Moreover, the number of elements in each component is the same and is a product of factorials.

Example

The following digraph \mathbb{G} has Maltsev, join and meet semilattice polymorphisms.



It has only four endomorphisms: id , 0 , 1 and inversion, they are all isolated. However, id is connected to 0 in $\mathbb{G}^{\mathbb{G}}$:

$$\text{id} = x \wedge 1 \rightarrow x \wedge a \rightarrow x \wedge 0 = 0.$$

Theorem (M, Zádori; 2012)

If \mathbb{G} is a connected reflexive digraph with Hobby-McKenzie polymorphisms, then $\text{End}(\mathbb{G})$ is connected.

Theorem (Gyenezse; 2013)

Suppose, that $|\mathbb{G}| \geq 6$. Then $\mathbb{G}^{\mathbb{G}}$ is connected if and only if

- \mathbb{G} is empty,
- there exists $a \in G$ such that $a \rightarrow x$ for all $x \in G$, or
- there exists $a \in G$ such that $x \rightarrow a$ for all $x \in G$.

Theorem (Gyenezse; 2013)

If $|\mathbb{G}| \geq 6$ and $\text{Sym}(\mathbb{G})$ is connected, then $\mathbb{G}^{\mathbb{G}}$ must be also connected.

THE COMPONENT OF THE IDENTITY

Definition

A map $f \in \mathbb{G}^{\mathbb{G}}$ is **idempotent**, if $f^2 = f$, it is a **retraction**, if $f \rightarrow f$ and $f^2 = f$, and it is **proper**, if $f \neq \text{id}$.

Lemma (M, Zádori; 2012)

If \mathbb{G} is reflexive or symmetric and the component of the identity in $\text{End}(\mathbb{G})$ contains something other than id , then it contains a proper retraction.

Theorem

If the smooth component of id in $\mathbb{G}^{\mathbb{G}}$ (or in any submonoid) contains a non-permutation, then it contains a proper retraction.

Proposition

If \mathbb{G} is smooth and the component of id contains a constant map, then the smooth part of $\mathbb{G}^{\mathbb{G}}$ is connected (and \mathbb{G} is connected and contains a loop).

THE COMPONENT OF THE IDENTITY

Example

The digraph $\mathbb{G} = (\{0, 1, 2\}; \neq)$ with 6 edges is connected, smooth, has algebraic length 1, and the identity in $\mathbb{G}^{\mathbb{G}}$ is isolated.

Example

Let $\mathbb{H} = (H; E)$ be the example with Maltsev, join and meet morphisms:

$$H = \{0, 1\}^2 \quad \text{and} \quad E = \{(x, y, u, v) \in H^2 \mid y = u\}.$$

Then the component of the identity for $\mathbb{G} \times \mathbb{H}$ is non-trivial (isomorphic to \mathbb{H}), but it does not contain a non-permutation.

Problem

Find a nontrivial smooth, connected, algebraic length 1 digraph with Taylor polymorphism for which id is isolated in $\mathbb{G}^{\mathbb{G}}$.

UNARY POLYNOMIALS OF \mathbb{G}

Definition

$\text{Pol}_1(\mathbb{G})$ is the induced subgraph of $\mathbb{G}^{\mathbb{G}}$ on the set of **unary polynomials** of the algebra $\mathbf{G} = (G; \text{Hom}(\mathbb{G}))$.

Proposition

- $\text{Pol}_1(\mathbb{G}) \leq \mathbf{G}^{\mathbb{G}}$ is generated by the identity and the constant maps
- \mathbb{G} is an induced subgraph of $\text{Pol}_1(\mathbb{G})$ on the set of constant maps
- $\text{Pol}_1(\mathbb{G})$ is smooth if and only if \mathbb{G} is smooth
- If \mathbb{G} is smooth, connected and algebraic length 1, then every component of $\text{Pol}_1(\mathbb{G})$ has algebraic length 1

Proof.

For a polynomial $p = t(x, a_1, \dots, a_n)$ we can find an oriented cycle in \mathbb{G}^n of algebraic length 1 going through (a_1, \dots, a_n) . Then the polymorphism $t \in \text{Hom}(\mathbb{G}^{n+1}, \mathbb{G}) = \text{Hom}(\mathbb{G}^n, \mathbb{G}^{\mathbb{G}})$ maps this cycle to a cycle in $\text{Pol}_1(\mathbb{G})$.

Proposition

If \mathbb{G} is smooth, connected and algebraic length 1, then the connectedness relation on $\text{Pol}_1(\mathbb{G})$ is a congruence.

Definition

Let \mathbf{A} be an algebra. Two unary polynomials $p, q \in \text{Pol}_1(\mathbf{A})$ are **twins**, if there exists a term t of arity $n + 1$ and constants $\bar{a}, \bar{b} \in A^n$ such that

$$p = t(x, \bar{a}) \quad \text{and} \quad q = t(x, \bar{b}).$$

The transitive closure of twin polynomials is the **twin congruence** τ of the algebra $\text{Pol}_1(\mathbf{A})$.

Corollary

If \mathbb{G} is smooth, connected and algebraic length 1, then the twin congruence blocks are connected.

Theorem

If \mathbb{G} is smooth, connected and algebraic length 1, and the corresponding algebra $\mathbf{G} = (G; \text{Hom}(\mathbb{G}))$ generates a congruence join semi-distributive variety (omits types **1**, **2** and **5**), then $\text{Pol}_1(\mathbb{G})$ is connected.

Proof.

- for $a \in G$ let η_a be the projection kernel of $\text{Pol}_1(\mathbb{G})$ onto its a -th coordinate
- for any $a \in G$ and $p, q \in \text{Pol}_1(\mathbb{G})$ we have $p \eta_a p(a) \tau q(a) \eta_a q$, so $\tau \vee \eta_a = 1$
- use join semi-distributivity

$$\tau \vee \alpha = \tau \vee \beta \implies \tau \vee \alpha = \tau \vee (\alpha \wedge \beta)$$

to derive $\tau \vee (\bigwedge_a \eta_a) = 1$, that is $\tau = 1$. □

STRUCTURE OF $\text{Pol}_1(\mathbb{G})$

Problem

Let \mathbb{G} be smooth, connected, algebraic length 1 and with Taylor polymorphism. Describe the structure of $\text{Pol}_1(\mathbb{G})$ modulo connectivity.

- We can assume that all twins of the identity are permutations
- The component of the identity has compatible join and meet
- From the loop lemma, every component that contains an idempotent has a loop, that is a proper retraction.

Theorem

If \mathbb{G} is smooth, connected, algebraic length 1 and $\mathbf{G} = (G; \text{Hom}(\mathbb{G}))$ generates a congruence modular variety, then $\text{Pol}_1(\mathbb{G})$ is connected.

Conjecture

If \mathbb{G} is smooth, connected, algebraic length 1 and has Hobby-McKenzie polymorphisms for omitting types **1** and **5**, then $\text{Pol}_1(\mathbb{G})$ is connected.

CONNECTIVITY IN $\text{Pol}_2^{\text{id}}(\mathbb{G})$

Theorem (M, Zádori; 2012)

If \mathbb{G} is reflexive, connected and has Gumm polymorphisms, then π_1 and π_2 are connected in the graph $\text{Hom}^{\text{id}}(\mathbb{G}^2, \mathbb{G})$ of idempotent binary morphisms.

Theorem

If \mathbb{G} is a smooth, connected, algebraic length 1 digraph which has Gumm polymorphisms, then the digraph $\text{Pol}_2^{\text{id}}(\mathbb{G})$ on the set of idempotent binary polynomials of \mathbb{G} is connected (π_1 and π_2 are connected).

Proof.

Take a path $\text{id} = f_0 \sim f_1 \sim \dots \sim f_k = c$ in $\text{Pol}_1(\mathbb{G})$ for some constant c .

$$\begin{aligned} d_i(x, x, y) &= d_i(x, f_0(x), y) \sim d_i(x, f_1(x), y) \sim \dots \sim d_i(x, f_k(x), y) \\ &= d_i(x, c, y) = d_i(x, f_k(y), y) \sim \dots \sim d_i(x, f_0(y), y) = d_i(x, y, y), \text{ and} \\ p(x, y, y) &= p(f_0(x), f_0(y), y) \sim p(f_1(x), f_1(y), y) \sim \dots \sim p(c, c, y) = y. \end{aligned}$$

Definition

An induced subgraph \mathbb{K} of $\mathbb{G}^{\mathbb{H}}$ is an **idempotent \mathbb{G} -subalgebra**, if K is closed under the idempotent polynomials of \mathbb{G} .

(Connection to CD absorption...)

Proposition

If $\text{Pol}_2^{\text{id}}(\mathbb{G})$ is connected then every smooth idempotent subalgebra of $\mathbb{G}^{\mathbb{H}}$ is connected.

Theorem

If a strongly connected, algebraic length 1 digraph has Gumm polymorphisms, then \mathbb{G} is extremely connected (connected with double edges).

- Can we do something similar for arbitrary relational structures? What are the right notions of smoothness and algebraic length 1?
- Combinatorial vs. algebraic arguments
- We do not even have a complete connectivity description for reflexive or symmetric digraphs...
- How can we adapt the absorption work of Barto and Kozik from the context of $\mathbf{R} \leq \mathbf{A} \times \mathbf{B}$?
- Describe absorption in tame congruence theoretic terms.
- Relations to CSP: consistent set of maps, preserving solutions, maximal absorbing subuniverses vs. maximal idempotents, etc.

Thank You!