Radical Theory in General Algebra

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Largest Nilpotent Ideal
Setting: Finite Dimensional Algebras

- N is the largest nilpotent ideal of A - the "nilradical"
- A/N is a direct sum of simple algebras - "semisimple"

Nilradical

Setting: Commutative Rings

- $\mathcal{N}(R) = \Sigma\{I \lhd R | \forall i \in I \exists n \in \mathbb{Z}^+ : i^n = 0\}$
- Radical of an ideal
  - $\sqrt{I} = \{a \in R | \exists n \in \mathbb{Z}^+ : a_n \in I\}$
  - $\mathcal{N}(R/I) = \sqrt{I}/I$

Nilradical

Setting: Commutative Rings

- $\mathcal{N}(R) = \bigcap\{P | P \text{ a prime ideal of } R\}$
- $R/\mathcal{N}(R)$ is a subdirect product of prime rings - "semiprime"

Jacobson Radical
Setting: Associative Rings

- $a \circ b = a + b + ab$
- R is *quasiregular* if $(R, \circ)$ is a group
- $\mathcal{J}(R)$ is the largest ideal of R which is a quasiregular ring

Jacobson Radical

Setting: Associative Rings

- $\mathcal{N}(R) \subseteq \mathcal{J}(R)$
- $R/\mathcal{J}(R)$ is a subdirect product of primitive rings - "semiprimitive"

Kurosh-Amitsur Radicals

Setting: Associative Rings

Radical class $\mathcal{R}$

i. If $I \triangleleft R \in \mathcal{R}$ then $R/I \in \mathcal{R}$

ii. $\mathcal{R}(R) = \Sigma\{I \triangleleft R | I \in \mathcal{R}\} \in \mathcal{R}$

iii. $\mathcal{R}(R/\mathcal{R}(R)) = 0$

Kurosh-Amitsur Radicals

Setting: Associative Rings

Semisimple Class $\mathcal{S}$

- $\mathcal{S} = \{R | \mathcal{R}(R) = 0\}$
- $\mathcal{S}$ is closed under ideals, subdirect products and extensions

Kurosh-Amitsur Radicals

Setting: Associative Rings

Special Radicals

- If $R \in \mathcal{S}$ then R is a subdirect product of rings from $\mathcal{M}$, where $\mathcal{M}$ is a "special class" of prime rings

Hoehnke Radicals
Setting: General Algebra!

$\rho : \{\text{algebras}\} \to \{\text{congruences}\}$, $A \mapsto \rho A$

- if f is a homomorphism on A, $f(\rho A) \subseteq \rho(f(A))$
- $\rho(A/\rho A)$ is always the identical relation

ρ is a *Hoehnke radical* and $\mathbb{S}_\rho = \{A \mid \rho A \text{ is the identical relation}\}$ is the *semisimple class* of ρ

Hoehnke Radicals
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Hoehnke Radicals

Setting: General Algebra

If $\mathcal{S}$ is a subdirectly closed class of algebras,

$$\rho_{\mathcal{S}} : A \mapsto \bigcap \{\sigma \mid A/\sigma \in \mathcal{S}\}$$

is a Hoehnke radical, so Hoehnke semisimple classes are exactly subdirectly closed classes

$$\mathbb{R}_\rho = \{A \mid \rho A \text{ is the universal relation}\}$$

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Hoehnke Radicals
Setting: General Algebra

If congruences are defined by distinguished subalgebras (so ρA "is" a subalgebra) and

- $\rho(\rho A) = \rho A$ and
- if $\rho B = B$ and $B \triangleleft A$ then $B \subseteq \rho A$

then ρ is a Kurosh-Amitsur radical

Radical Operations
Setting: Ω-groups

A closure operation C on ideals of Ω-groups R, S satisfying

- If $I/K \cong C(J)/J$ then $I \subseteq C(J)$

for $I, K \triangleleft R, J \triangleleft S$ is a *radical operation*; $\mathcal{R} = \{C(A)/A\}$ is a Kurosh-Amitsur radical class

Torsion Theory
Setting: Ω-groups

A pair of classes $\mathcal{X}, \mathcal{Y}$ satisfying

- $\mathcal{X} \cap \mathcal{Y} = 0$
- $\mathcal{X}$ is homomorphically closed
- $\mathcal{Y}$ is ideal closed
- For every Ω-group a there is an Ω-group b with $b \in \mathcal{X}$ and $a/b \in \mathcal{Y}$

Kurosh-Amitsur Radicals again

Setting: 0-normal Subtractive Varieties

Every algebra has a 0; for any congruence ρ the ρ-class of 0 determines ρ; for any operation σ, $\sigma(0, \ldots, 0) = 0$

Radical Class

1. For $I, J \lhd A$, $I \in \mathcal{R}$ there is a $K \lhd A/J$ with $(I \vee J)/J \subseteq K \in \mathcal{R}$
2. Every algebra A has an ideal $\mathcal{R}(A) \in \mathcal{R}$ containing all the $\mathcal{R}$-ideals of A
3. $\mathcal{R}(A/\mathcal{R}(A)) = 0$

Kurosh-Amitsur Radicals again

Setting: 0-normal Subtractive Varieties

Semisimple Class

1. $0 \in \mathcal{S}$
2. $I \triangleleft A$ and $A/I \in \mathcal{S}$ imply that for all $J \triangleleft A$ for which $J \not\subseteq I$, J has a proper ideal K such that $J/K \in \mathcal{S}$;
3. $\mathcal{S}$ is closed under subdirect products; and
4. $A(\mathcal{S})(\mathcal{S}) = A(\mathcal{S})$ for every A.

where $A(\mathcal{S}) = \bigcap\{I \triangleleft A \mid A/I \in \mathcal{S}\}$

Puczyłowski Framework

Class $\mathcal{A}$ of *algebras*, $0 \in \mathcal{A}$, $\cong$ *isomorphism*

i. For $a \in \mathcal{A}$ a complete lattice $(L_a, \leq) \subseteq \mathcal{A}$, with top and bottom elements a and 0 respectively; $i \in L_a$ is an *ideal* of a $(i \lhd a)$, $i \lhd \ldots \lhd a$ is an *accessible*

ii. For $i \in L_a$, the interval $[0, i]_{L_a} = \{b \in L_a \mid 0 \leq b \leq i\}$ is a sublattice of L_i

iii. With every $i \lhd a$, there exists $a/i \in \mathcal{A}$, with $L_{a/i} = \{k/i \mid k \lhd a, i \leq k \leq a\}$, and the map given by $k \mapsto k/i$ is an isomorphism $[i, a]_{L_a} \to L_{a/i}$; a/i is a *factor* of a

iv. If $a \cong b$ then there is an isomorphism $f : L_a \to L_b$ such that for each $i \in L_a$, $i \cong f(i)$ and $a/i \cong b/f(i)$

v. For each $a \in \mathcal{A}$ and $i, j \lhd a$ with $j \leq i$, $(a/j)/(i/j) \cong a/i$, and for any $i, j \lhd a$, $(i \vee j)/i \cong j/(i \wedge j)$

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Base Radicals
Setting: Puczyłowski Framework

For any class $\mathcal{X}$

- $\mathbf{U}(\mathcal{X}) = \{a \,|\, a \text{ has no factor in } \mathcal{X}\}$
- $\mathbf{S}(\mathcal{X}) = \{a \,|\, a \text{ has no accessible in } \mathcal{X}\}$

A class $\mathcal{R}$ is a *base radical class* if $\mathbf{US}(\mathcal{R}) = \mathcal{R}$

A class $\mathcal{S}$ is a *base semisimple class* if $\mathbf{SU}(\mathcal{S}) = \mathcal{S}$

Base Radicals

Setting: Puczyłowski Framework

For any $\mathcal{X}$

- **US**$(\mathcal{X})$ is the base radical class of $\mathcal{X}$
- **SU**$(\mathcal{X})$ is the base semisimple class of $\mathcal{X}$

For factor closed $\mathcal{X}$, $\mathcal{X} \subseteq$ **US**$(\mathcal{X})$

For accessibly closed $\mathcal{X}$, $\mathcal{X} \subseteq$ **SU**$(\mathcal{X})$

Kurosh-Amitsur Radicals, yet again

Setting: Puczyłowski Framework

Define

$$\mathbf{S}'(\mathcal{X}) = \{a \mid a \text{ has no nonzero } \mathcal{X}\text{-ideal}\}$$

Then $\mathcal{R}$ is a KA-radical if $\mathbf{US}'(\mathcal{R}) = \mathcal{R}$ (Puczyłowski) and base radicals are the KA-radicals which have accessibly closed semisimple classes

All Of Them

Setting: Puczyłowski Framework

torsion theories $\subseteq$ base radicals $\subseteq$ KA-radicals

For $\mathcal{R}$ a base radical class

$$\mathcal{R}(a) = \bigwedge\{i \triangleleft a \mid n \leq i \text{ for } \mathcal{R}\text{-accessibles of } a\}$$

$a \mapsto a/\mathcal{R}(a)$ is a Hoehnke radical