Topological Graph Inverse Semigroups

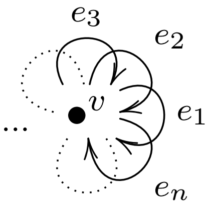
J. D. Mitchell

School of Mathematics and Statistics, University of St Andrews

University of St Andrews

The big idea

Given a **directed graph** E, we construct an **inverse semigroup**, whose elements are paths in E, and we **topologize** it.



For example, the **polycyclic monoid** P_n, is defined (by Nivat-Perrot) as:

$$P_n = \langle e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1} : e_i^{-1} e_j = \delta_{ij} \rangle,$$

where δ_{ij} is the **Kronecker delta**, and P_1 is the **bicyclic monoid** (with a zero adjoined).

Why should I care...

...about polycyclic monoids?

A **positive** $w \in P_n$ is a product of positive powers of $e_1, \ldots, e_n$.

Theorem (Meakin-Sapir '93)

There is a lattice isomorphism between the lattice of congruences on the free monoid $\{e_1, \ldots, e_n\}$ *and the lattice of submonoids* U *of* P_n *with the property that* $pUp^{-1} \subseteq U$ *for every positive* $p \in P_n$*.*

Corollary (The most famous open problem in semigroup theory)

The word problem for 1-relation semigroups is equivalent to the membership problem for 1-generator submonoids of P_n*.*

More advertising...

The bicyclic monoid P_1 is an ubiquitous example in semigroup theory.

Polycyclic monoids have lots of interesting properties they are/can used to:

- congruence-free
- the inverse monoids generated by the pops and pushes of pushdown stacks
- recognise context-free languages
- the syntactic monoids of correct bracketing languages

Graph inverse semigroups:

- generalizing polycyclic monoids
- arise in the study of rings and C^*-algebras, specifically in the study of Cohn path and Leavitt path algebras.

What is...

...a topological semigroup?

If S is a semigroup with a topology defined on it, then S is a **topological semigroup** if multiplication

$$(x, y) \mapsto xy$$

is continuous.

If S is an inverse semigroup that is a topological semigroup, then S is a **topological inverse semigroup** if the inverse operation

$$x \mapsto x^{-1}$$

is continuous.

For example, every semigroup is a topological semigroup w.r.t. the discrete or the trivial topology.

Why should I care...

...about topological semigroups?

Answer: you can use topology to prove theorems about your semigroup.

The symmetric group $\mathrm{Sym}(\mathbb{N})$ is a topological group w.r.t. the complete metric:

$$d(f, g) = \frac{1}{m} + \frac{1}{n}$$

where

$$m = \min\{\, i \in \mathbb{N} \,:\, (i)f \neq (i)g \,\} \text{ and } n = \min\{\, i \in \mathbb{N} \,:\, (i)f^{-1} \neq (i)g^{-1} \,\}.$$

Theorem (Ore '51 in 7 pages)

Every element of $\mathrm{Sym}(\mathbb{N})$ *is a commutator.*

Proof. A simple application of Baire's Category Theorem. □

Topologizing the bicyclic monoid

The **bicyclic monoid:**

$$B = \langle b, c \mid bc = 1 \rangle = P_1 \setminus \{0\}.$$

Theorem (Eberhart-Selden '69)

The bicyclic monoid has a unique non-trivial Hausdorff semigroup topology: the discrete topology.

Theorem (Eberhart-Selden '69)

If B is dense in a Hausdorff topological semigroup S, then:

- *B is open and discrete in S;*
- *$S \setminus B$ is an ideal in S.*

If S is a Hausdorff topological inverse semigroup, then

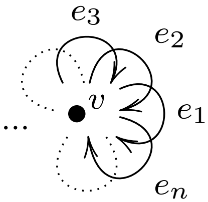
- *$S \setminus B$ is a group with a dense cyclic subgroup (monothetic).*

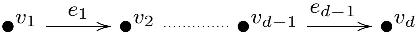
What is...

...a graph?

A **directed graph** E consists of **vertices** E^0, **edges** E^1, **range** function $\mathbf{r} : E^1 \longrightarrow E^0$, and **source** function $\mathbf{s} : E^1 \longrightarrow E^0$.

A **path** x in E is a finite sequence of edges $x = e_1 \cdots e_n$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, \ldots, n-1$.





What is...

...a graph inverse monoid?

If $E = (E^0, E^1, \mathbf{r}, \mathbf{s})$ is a graph, then the **graph inverse semigroup $G(E)$ of E** is the semigroup with zero generated by

$$E^0 \text{ and } E^1 \cup \{e^{-1} : e \in E^1\}$$

satisfying the relations:

$$\begin{aligned}
\text{(V)} &\quad v \cdot w = \delta_{v,w} v, \\
\text{(E1)} &\quad \mathbf{s}(e) \cdot e = e \cdot \mathbf{r}(e) = e, \\
\text{(E2)} &\quad \mathbf{r}(e) \cdot e^{-1} = e^{-1} \cdot \mathbf{s}(e) = e^{-1}, \\
\text{(CK1)} &\quad e^{-1} \cdot f = \delta_{e,f} \cdot \mathbf{r}(e).
\end{aligned}$$

for all $v, w \in E^0$ and $e, f \in E^1$.

Every nonzero element of $G(E)$ can be written uniquely as xy^{-1} for some $x, y \in G(E)$.

Topological graph inverse monoids

Theorem (Mesyan-M-Morayne-Péresse '13)

Suppose that E is a graph, and that $G(E)$ is a Hausdorff topological semigroup. Then $G(E) \setminus \{0\}$ is discrete.

Theorem (Mesyan-M-Morayne-Péresse '13)

Let E be a graph, and suppose that $G(E)$ is a dense subsemigroup of a Hausdorff topological semigroup S. Then

(1) *$G(E) \setminus \{0\}$ is open in S.*

(2) *$(S \setminus G(E)) \cup \{0\}$ is an ideal of S.*

Example (**We cannot weaken Hausdorff to T_1**)

Let E be the graph:

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \xrightarrow{e_3} \cdots$$

Then the topology generated by $\{0\}$ and

$$U_{pq^{-1},n} = \{pxx^{-1}q^{-1} : x \in \text{Path}(E),\ \mathbf{s}(x) = \mathbf{r}(p) = \mathbf{r}(q),\ |x| > n\} \cup \{pq^{-1}\}$$

makes $G(E)$ into a T_1 topological semigroup, and $G(E) \setminus \{0\}$ is not discrete.

Non-discrete topologies??

Proposition (Mesyan-M-Morayne-Péresse '13)

Let E be a graph having paths of arbitrary (finite). Then $\exists$ a non-discrete metrizable semigroup topology on $G(E)$.

Corollary (Mesyan-M-Morayne-Péresse '13)

Let E be a finite graph with at least one cycle. Then $\exists$ a metrizable topology on $G(E)$ such that $G(E)$ is closed in any Hausdorff topological semigroup S containing $G(E)$.

Theorem (Mesyan-M-Morayne-Péresse '13)

If E is a finite graph, then the only locally compact Hausdorff semigroup topology on $G(E)$ is the discrete topology.

Cardinality of the complement

Polycyclic monoids

Proposition (Mesyan-M-Morayne-Péresse ’13)

Let $n \geq 2$, and suppose that P_n is a subsemigroup of a Hausdorff topological semigroup. Then $\overline{P_n} \setminus P_n$ is either empty or infinite.

Example

There exist metrizable semigroups S such that:

- P_1 is dense in S and $|S \setminus P_1| = 1$;
- $|S \setminus P_n| = \aleph_0$, and P_n is not discrete for all $n > 2$;
- P_2 is dense in S, $|S \setminus P_2| = 2^{\aleph_0}$, P_2 is discrete.