Varieties of Abelian Topological Groups with Coproducts

by

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Varieties of groups were introduced in the 1930s by Garret Birkhoff and Bernhard Neumann. A variety of groups can be described as a class of groups satisfying a certain class of laws.

For example the variety of all abelian groups is the class of groups satisfying the law $a^{-1}b^{-1}ab = 1$.

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For example the variety of all abelian groups is the class of groups satisfying the law $a^{-1}b^{-1}ab = 1$.

A variety of groups can also be defined as follows:

Definition 1. A **variety of groups** is a class of groups closed under the operations of forming Quotient groups (Q), Subgroups (S) and arbitrary Cartesian product groups (C).

The book “Varieties of Groups” by Hanna Neumann, Springer-Verlag, 1967 has the easily verified result:

Proposition 1. If $\boldsymbol{\Omega}$ is any non-empty class of groups and $\boldsymbol{V}(\boldsymbol{\Omega})$ is the smallest variety of groups containing $\boldsymbol{\Omega}$, then $\boldsymbol{V}(\boldsymbol{\Omega}) = \boldsymbol{QSC}(\boldsymbol{\Omega})$.

This proposition tells us that every member of $V(\Omega)$ can be written as a quotient group of a subgroup of some cartesian product of members of Ω.

It is significant that each of the operations (Q), (S) and (C) needs to be used only once.

For a non-empty family $\{G_i\}_{i\in I}$ of abelian groups, the **direct sum** of G_i is denoted by $\bigoplus_{i\in I} G_i$ and is

$$\{(g_i)_{i\in I} \in \prod_{i\in I} G_i \colon g_i = 1, \forall \text{ but a finite number of } i\}.$$

The group $\bigoplus_{i\in I} G_i$ is the **coproduct** of the family $\{G_i\}_{i\in I}$ in the category, $\mathbf{Ab}$, of all abelian groups and group homomorphisms.

Note that $\bigoplus_{i\in I} G_i$ is a subgroup of the cartesian product $\prod_{i\in I} G_i$.

Definition 2. Let $FA(X)$ an abelain group containing a set X. Then $FA(X)$ is said to be the **free abelian group** on X if for each mapping ϕ of X into each abelian group G, there is a unique homomorphism Φ of $FA(X)$ into G such that $\Phi|X = \phi$.

The free abelian group on an arbitrary set is isomorphic to $\bigoplus_{i \in I} \mathbb{Z}_i$ for some index set I, where $\mathbb{Z}_i$ is the additive group of integers.

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Proposition 2. Every abelian group G is a quotient of the free abelian group on the underlying set of G.

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For each prime number p, the class of abelian groups satisfying the law $x^p = 1$ is a variety, and distinct primes, p, result in distinct varieties. So there is certainly an infinite number of distinct varieties of abelian groups. And it is known that there are precisely a countably infinite number of varieties of abelian groups.

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(iii) the number of distinct varieties of abelian groups is $\boldsymbol{\aleph_0}$;

(iv) $\boldsymbol{V}(\boldsymbol{\Omega})$ is closed under the operation $(\boldsymbol{K})$ of forming the coproduct in the category of all abelian groups and group homomorphisms.

In 1969 the third author (Sidney A. Morris) introduced the notion of a **variety of topological groups** defined to be a non-empty class $\mathfrak{V}$ of (not necessarily Hausdorff) topological groups which is closed under the operations of forming subgroups $\boldsymbol{S}$, (not necessarily Hausdorff) quotient topological groups $\boldsymbol{Q}$ and arbitrary cartesian products $\boldsymbol{C}$ (with the Tychonoff product topology).

For example, the class $\mathbf{TopAb}$ of all abelian topological groups is a variety of topological groups.

Walter Taylor (while in Australia on a Fulbright Fellowship) proved that such varieties can also be defined using contingent limit laws and this was followed up by Ralph Kopperman, Mike W. Mislove, Sidney A. Morris, Peter Nickolas, Vladimir Pestov, and Sergey Svetlichny.

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Limit laws, let alone contingent limit laws, are not terribly easy to describe. But one example might assist: The topological group $\mathbb{Q}/\mathbb{Z}$ satisfies the limit law

$$n!x \to 0, \text{ as } n \to \infty, \text{ for each } x \in \mathbb{Q}/\mathbb{Z}.$$

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The varieties generated by the most important classes of topological groups, including the class of locally compact abelian groups, Banach spaces, groups having a subgroup topology, Lie groups, and SIN groups, were studied during the last 40 years by Malcolm Brooks, Su-shing Chen, Joe Diestel, Karl H. Hofmann, Arkady Leiderman, Sidney A. Morris, Vladimir Pestov, Steve Saxon, Carolyn McPhail Sandison, and Markus Stroppel.

Let us summarize a few of the most important results on varieties of topological for our purposes.

Note that the operation $\boldsymbol{K}$ denotes that of forming of the coproduct in the category of all abelian topological groups and continuous homomorphisms.

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The operation of forming finite products is denoted by $\boldsymbol{P}$.

The class of all (i) discrete abelian topological groups, (ii) locally compact abelian topological groups, (iii) topological groups underlying Banach spaces, and (iv) σ-compact abelian topological groups are respectively denoted by (i) $\mathcal{D}$, (ii) $\mathcal{LCA}$, (iii) $\mathcal{B}$, and (iv) C_σ.

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(vi) $\mathfrak{V}(\prod_{n\in\mathbb{N}} \boldsymbol{FA}([0,1]^n)) \subsetneqq \mathfrak{V}(\boldsymbol{C_\sigma}) \subsetneqq \mathfrak{V}(\boldsymbol{\ell_1}) \subsetneqq \mathbf{TopAb}$;

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(vii) If $\boldsymbol{p} \neq \boldsymbol{q}$, $\boldsymbol{p}, \boldsymbol{q} \geq 1$, then $\boldsymbol{\ell_p} \notin \mathfrak{V}(\boldsymbol{\ell_q})$;

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(vii) If $p \neq q$, $p, q \geq 1$, then $\ell_p \notin \mathfrak{V}(\ell_q)$;

(viii) $\mathfrak{V}(\mathcal{B}) = \mathbf{TopAb}.$

We saw a surprising result, namely there is a proper class of varieties. To see this we introduce T(m)-groups.

Definition 3. Let m be an infinite cardinal number. The topological group G is said to be a **$\mathbf{T}(m)$-group** if each neighbourhood U of the identity of G contains a subgroup H such that the index of H in G is strictly less than m. (For abelian groups this is equivalent to cardinality of G/H $< m$.)

It is clear that if G is a topological group of cardinality strictly less than m, then G is a $\mathsf{T}(m)$-group.

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For each cardinal m, the class of all $\mathsf{T}(m)$-groups is a variety and distinct cardinal numbers results in distinct varieties of topological groups.

So there is a proper class of varieties of topological groups. Further, the variety of all abelian topological groups is not singly generated.

Definition 4. (i) The filter of all open neighborhoods of the identity 1 of a topological group (G, τ) is denoted by $\mathcal{N}(G)$.

(ii) Let $\{G_i\}_{i \in I}$ be a non-empty family of groups. The natural projection of $\prod_{i \in I} G_i$ onto G_k is denoted by π_k.

(iii) The natural inclusion of G_k into $\bigoplus_{i \in I} G_i$ is denoted by j_k.

Definition 5. Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of abelian topological groups. For every $i \in I$ fix $U_i \in \mathcal{N}(G_i)$ and put

$$\prod_{i\in I} U_i := \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i : \ g_i \in U_i \text{ for all } i \in I \right\}.$$

Then the sets of the form $\prod_{i\in I} U_i$, where $U_i \in \mathcal{N}(G_i)$ for every $i \in I$, form a neighbourhood basis at the identity of a topological group topology $\mathcal{T}_b$ on $\prod_{i\in I} G_i$ that is called the **box topology**.

Definition 6. The **coproduct topology $\mathcal{T}_f$** on $G = \bigoplus_{i \in I} G_i$ coincides with **the final group topology** with respect to the family of canonical homomorphisms $j_k : G_k \to G$, where the final group topology is the finest group topology on G such that all j_k are continuous. Then all j_k are embedding into $(G, \mathcal{T}_f)$ and the group $(G, \mathcal{T}_f)$ is called the **coproduct** of the family $\{(G_i, \tau_i)\}_{i \in I}$.

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Clearly, $\mathcal{T}_b \leq \mathcal{T}_f$ on $\bigoplus_{i \in I} G_i$. It follows that the coproduct of any family of discrete abelian topological groups has a finer topology than the box topology and so is discrete.

Proposition 4. Let Ω be any family of abelian topological groups. Then

(i) $QQ(\Omega) = Q(\Omega)$;

(ii) $SQ(\Omega) = QS(\Omega)$;

(iii) $CC(\Omega) = C(\Omega)$;

(iv) $PP(\Omega) = P(\Omega)$;

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(viii) $SS(\Omega) = S(\Omega)$;

(ix) $CQ(\Omega) \subseteq QC(\Omega)$;

(x) $CS(\Omega) \subseteq SC(\Omega)$. ∎

Proposition 5. Let Ω be an any family of abelian topological groups. Then

(i) $\boldsymbol{KQ}(\Omega) \subseteq \boldsymbol{QK}(\Omega)$;

(ii) $\boldsymbol{KS}(\Omega) \subseteq \boldsymbol{SK}(\Omega)$.

Proposition 6. Let Ω be any family of abelian topological groups. Then

$$\boldsymbol{QSCK.QSCK}(\boldsymbol{\Omega}) \subseteq \boldsymbol{QS}[\boldsymbol{CKCK}](\boldsymbol{\Omega}).$$

Proof. Using Propositions 4 & 5, we have

$$\begin{aligned} QSC(K.Q)SCK &\subseteq Q(SCQ)(KS)CK \\ &\subseteq Q(QSC)(SK)CK \\ &\subseteq QS(CS)KCK \\ &\subseteq QSSCKCK \\ &= QS[CKCK]. \end{aligned}$$

■

As a corollary of Proposition 6 we obtain:

Theorem 3. For any class $\mathbf{\Omega}$ of abelian topological groups, the variety, $\mathfrak{C}(\mathbf{\Omega})$, of topological groups with coproducts generated by $\mathbf{\Omega}$ is given by

$$\bigcup_{n\in\mathbb{N}} \boldsymbol{QS}\left[\boldsymbol{C}_1\boldsymbol{K}_1.\boldsymbol{C}_2\boldsymbol{K}_2\ldots\boldsymbol{C}_n\boldsymbol{K}_n\right](\mathbf{\Omega}).$$

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Let $\mathcal{S}$ denote the class of all abeilan topological groups with a subgroup topology, that is, a basis of neighbourhoods of the identity being subgroups.

Proposition 7.

(i) $\mathfrak{C}(\mathbb{Z}) = \mathfrak{C}(\mathcal{D}) = \mathcal{S}$.

(ii) $\mathbb{Q} \notin \mathfrak{C}(\mathbb{Z})$, $\mathbb{T} \notin \mathfrak{C}(\mathbb{Z})$, and $\mathbb{R} \notin \mathfrak{C}(\mathbb{Z})$. ■

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Proposition 8.

(i) $\mathfrak{C}(\mathbb{T}) \subsetneqq \mathfrak{C}(\mathbb{T}, \mathbb{Z})$;

(iii) $\mathfrak{C}(\mathbb{Z}) \subsetneqq \mathfrak{C}(\mathbb{T}, \mathbb{Z})$;

(iii) $\mathfrak{C}(\mathbb{T}, \mathbb{Z}) \subseteq \mathfrak{C}(\mathbb{R})$;

(ii) $\mathfrak{C}(\mathbb{R}) = \mathfrak{C}(\mathbb{R}, \mathcal{D}) = \mathfrak{C}(\mathcal{LCA})$.

The next result is surprising and says that $\mathbf{TopAb}$ as a variety with coproducts is singly generated.

Theorem 4. If $\mathfrak{s}$ is the convergent sequence $0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ and $FA(\mathfrak{s})$ is the free abelian topological group on the countable compact space $\mathfrak{s}$, then $\mathfrak{C}(FA(\mathfrak{s})) = \mathbf{TopAb}$. Indeed, if X is any completely regular Hausdorff space containing a convergent sequence of distinct points, then $\mathfrak{C}(FA(X)) = \mathbf{TopAb}$. So $\mathfrak{C}(FA([0,1])) = \mathbf{TopAb}$.

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(Clearly there are at least a countably infinite number.)

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Problem 5. Is $\mathfrak{C}(\mathcal{LCA}) = \mathbf{TopAb}$?

Key References

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Thank You