Principal Mappings Between Posets

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Outline of Talk

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Motivation

Let R be a commutative ring and Idl(R) be the lattice of ideals of R.

For each ideal I of R, define the mappings

F_I, G_I : Idl(R) $\rightarrow$ Idl(R) by

$$F_I(J) = IJ,$$

$$G_I(J) = [J : I] = \bigvee\{M \in \mathrm{Idl}(R) : IM \subseteq J\}, \; J \in \mathrm{Idl}(R),$$

where IJ is the multiplication of I with J.

If I is a **principal ideal** of R,

then for any J, K in $Idl(R)$,

$$(J \cap [K : I])I = JI \cap K \qquad (1)$$

$$[(K + JI) : I] = [K : I] + J \qquad (2)$$

In terms of F_I and G_I,

(1) becomes

$$F_I(J \cap G_I(K)) = F_I(J) \cap K,$$

and (2) becomes

$$G_I(K + F_I(J)) = G_I(K) + J.$$

- In 1960's, Dilworth introduced the **principal elements** in a multiplicative lattice (such as Idl(R)).

If a is a principal element of L, then for all x, $y \in L$,

$$(x \wedge [y : a])a = xa \wedge y,$$

$$[(y \vee xa) : a] = [y : a] \vee x.$$

- Dilworth used this to generalize most of the deep results on ideals of commutative Noetherian rings to Noether lattices, such as **Noether's Primary Decomposition Theorem** and the **Intersection Theorem**.

- The definition of Dilworth's principal elements involves the multiplication operation on lattices, and so does not apply to a general lattice.

- We defined principal mappings between lattices in our paper "A Generalization of Dilworth's Principal Elements" in 2012.

  From

$$F_I(J \cap G_I(K)) = F_I(J) \cap K,$$
$$G_I(K + F_I(J)) = G_I(K) + J,$$

  we define a mapping $f: L \to M$ between two lattices to be a **principal mapping** if there exists a mapping $g: M \to L$ such that for all $x \in L$, $y \in M$,

$$f(x \wedge g(y)) = f(x) \wedge y, \quad g(y \vee f(x)) = g(y) \vee x.$$

Principal Mapping Between Posets - Definition

A mapping $f : P \to Q$ between two posets P and Q is called a **principal mapping** if there is a mapping $g : Q \to P$ such that the following equations hold for all $x \in P$, $y \in Q$:

$$f(\downarrow x \cap \downarrow g(y)) = \downarrow f(x) \cap \downarrow y \,,$$
$$g(\uparrow y \cap \uparrow f(x)) = \uparrow g(y) \cap \uparrow x \,.$$

The mapping g is then called the **residual** of f.

Examples of Principal Mappings

- Every isomorphism between posets is a principal mapping.

- Let X be any set and $A \subseteq X$. Then $F_A : \mathscr{P}(X) \to \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the lattice of power sets of X, is a principal mapping where $\boldsymbol{F_A(B) = A \cap B}$ for any $B \in \mathscr{P}(X)$.

- Let L be a multiplicative lattice and $a \in L$. Define $\phi_a : L \to L$ by $\boldsymbol{\phi_a(x) = ax}$ for any $x \in L$. Then ϕ_a is a **principal mapping** if and only if a is a **principal element** of L.

Some General Properties

Let $f : P \to Q$ be a principal mapping between posets with $g : Q \to P$ as the residual. Then

- $(f\ , g)$ is a **Galois connection (residuated pair)** between P and Q. In particular, f and g are monotone.
- $f(P)$ is a **lower subset** of Q, that is,
$$\downarrow f(P) = f(P).$$
- $g(Q)$ is an **upper subset** of P, that is,
$$\uparrow g(Q) = g(Q).$$

Theorem 1 :

A mapping $f : P \to Q$ between two posets P and Q is a **principal mapping** if and only if

(i) f has an upper adjoint $g : Q \to P$; and

(ii) for all $x \in P$, $y \in Q$,

$$\downarrow f(x) = f(\downarrow x), \quad \uparrow g(y) = g(\uparrow y).$$

THEOREM 2 :

A mapping $f : L \to M$ between two lattices L and M is a **principal mapping** if and only if there is a mapping $g : M \to L$ such that the following equations hold for all $x \in L$, $y \in M$:

$$f(x \wedge g(y)) = f(x) \wedge y \, ,$$

$$g(y \vee f(x)) = g(y) \vee x \, .$$

Composition Of Principal Mappings

- If $f_1 : P_1 \to P_2$ and $f_2 : P_2 \to P_3$ are principal mappings between posets, then

  $f_2 \circ f_1 : P_1 \to P_3$ is a principal mapping.

The above property generalizes the result that **the product of two principal elements in a multiplicative lattice is still a principal element,** first proved by Dilworth in "Abstract Commutative Ideal Theory".

Let L be a multiplicative lattice.

According to Dilworth,
an element a of L is called
(i) a **meet principal element** if for all $x, y \in L$,

$$(x \wedge [y : a])a = xa \wedge y \,;$$

(ii) a **join principal element** if for all $x, y \in L$,

$$[(y \vee xa) : a] = [y : a] \vee x \,.$$

Meet Principal and Join Principal Mappings Between Posets

Let $f : P \to Q$ be a mapping between posets P and Q with $g : Q \to P$ as an upper adjoint.

(i) f is called a **meet principal mapping** if for all $x \in P$, $y \in Q$,

$$f(\downarrow x \cap \downarrow g(y)) = \downarrow f(x) \cap \downarrow y \, ;$$

(ii) f is called a **join principal mapping** if for all $x \in P$, $y \in Q$,

$$g(\uparrow y \cap \uparrow f(x)) = \uparrow g(y) \cap \uparrow x \, .$$

f is a principal mapping if and only if f is both a meet principal and a join principal mapping.

An Example of a Meet Principal Mapping

Let X be a topological space and $\mathcal{O}(X)$ be the complete lattice of open sets of X. Given any $U \in \mathcal{O}(X)$, define the mapping

$\phi_u : \mathcal{O}(X) \to \mathcal{O}(X)$ by

$$\boldsymbol{\phi_u(V) = U \cap V}, \; V \in \mathcal{O}(X).$$

Then ϕ_u is always **a meet principal mapping**, **that is not a join principal mapping**

unless U is closed.

Let L be a multiplicative lattice.

It has been defined in Abstract Ideal Theory that an element a of L is called

(i) a **weak meet principal element** if for all $y \in L$,

$$[y : a]a = a \wedge y\ ;$$

[Put $x = 1_L$ in $(x \wedge [y : a])a = xa \wedge y$.]

(ii) a **weak join principal element** if for all $x \in L$,

$$[xa : a] = [0_L : a] \vee x.$$

[Put $y = 0_L$ in $[(y \vee xa) : a] = [y : a] \vee x$.]

If P has a top element 1_P, putting $x = 1_P$ in

$$f(\downarrow x \cap \downarrow g(y)) = \downarrow f(x) \cap \downarrow y$$

gives

$$f(\downarrow 1_P \cap \downarrow g(y)) = \downarrow f(1_P) \cap \downarrow y$$

which can be shown to be equivalent to

$$f(\downarrow 1_P) = \downarrow f(1_P).$$

If Q has a bottom element 0_Q, putting $y = 0_Q$ in

$$g(\uparrow y \cap \uparrow f(x)) = \uparrow g(y) \cap \uparrow x$$

gives

$$g(\uparrow 0_Q \cap \uparrow f(x)) = \uparrow g(0_Q) \cap \uparrow x$$

which can be shown to be equivalent to

$$g(\uparrow 0_Q) = \uparrow g(0_Q).$$

Weak Principal Mappings Between Posets

Let $f : P \to Q$ be a mapping between posets P and Q with $g : Q \to P$ as an upper adjoint.

(i) f is called a **weak meet principal mapping** if

$$\downarrow f(P) = f(P).$$

(ii) f is called a **weak join principal mapping** if

$$\uparrow g(Q) = g(Q).$$

If f is both a weak meet principal and a weak join principal mapping, then f is called a **weak principal mapping**.

PROPOSITION 3 :

Let $f : P \to Q$ be a mapping between posets P and Q with $g : Q \to P$ as an upper adjoint.

(i) If P has a top element 1_P, then f is a **weak meet principal mapping** if and only if for all $y \in Q$,

$$f(g(y)) = f(1_P) \wedge y.$$

(ii) If Q has a bottom element 0_Q, then f is a **weak join principal mapping** if and only if for all $x \in P$,

$$g(f(x)) = g(0_Q) \vee x.$$

Examples of Weak Meet Principal and Weak Join Principal Mappings

Let L be a frame.

For any $a \in L$, define $\psi_a : L \to L$ by

$$\psi_a(x) = a \wedge x \text{ for all } x \in L.$$

Then ψ_a is always a **meet principal mapping.**

ψ_a is a **weak join principal mapping** if and only if a has a complement $a^{\perp}$ in L,

that is, there exists $a^{\perp} \in L$ such that

$$a \wedge a^{\perp} = 0_L \text{ and } a \vee a^{\perp} = 1_L$$

where 0_L and 1_L are the bottom and top elements of L respectively.

Examples of Weak Meet Principal and Weak Join Principal Mappings

Let $h : X \rightarrow Y$ be a mapping from set X to set Y. Define the mapping $\mathrm{F}_h : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ between the lattices of power sets of Y and X respectively by

$$\mathbf{F}_h(V) = h^{-1}(V)$$

for any $V \in \mathcal{P}(Y)$.

Then F_h is a **weak join principal mapping.**

F_h is a **weak meet principal mapping** if and only if h is injective.

The composition of two weak meet (weak join) principal mappings between posets need not be a weak meet (weak join) principal mapping.

PROPOSITION 4 :

The composition of a weak meet principal mapping followed by a meet principal mapping between posets is a weak meet principal mapping.

THEOREM 5 :

Let $f : L \to M$ be a mapping between **bounded modular lattices** with an upper adjoint.

Then f is a principal mapping if and only f is a weak principal mapping.

COROLLARY 6 :

An element in a modular multiplicative lattice L is principal if and only if it is weak principal.

(A result in several papers on Abstract Ideal Theory.)

THEOREM 7 :

Let $f : P \to Q$ be a monotone mapping between posets. Let $\mathcal{D}(P)$ and $\mathcal{D}(Q)$ be the lattices of all the lower sets of P and Q respectively.

Then $f^*: \mathcal{D}(P) \to \mathcal{D}(Q)$ defined by

$$f^*(A) = \downarrow f(A)$$

for each $A \in \mathcal{D}(P)$,

is a principal mapping if and only if f is an order embedding and satisfies

$$f(\downarrow a) = \downarrow f(a)$$

for all $a \in P$.

THEOREM 8 :

Let X, Y be two non-empty sets.

For any mapping $F : \mathscr{P}(X) \to \mathscr{P}(Y)$, the following statements are equivalent :

(1) F is a principal mapping.

(2) There is a set-valued mapping $h : X \to \mathscr{P}(Y)$ such that $F(A) = \cup \{h(x) : x \in A\}$ for each $A \in \mathscr{P}(X)$ and for any $x \in X$, $|h(x)| \leq 1$. Also, for any $x_1, x_2 \in X$, $h(x_1) = h(x_2)$ and $h(x_1) \neq \varnothing$ imply $x_1 = x_2$.

(3) There is a subset X_1 of X and an injective mapping $t : X_1 \to Y$ such that $F(A) = t(X_1 \cap A)$ for all $A \in \mathscr{P}(X)$.

Let L be a multiplicative lattice and $a \in L$.

The mapping $\phi_a : L \to L$ defined by

$$\boldsymbol{\phi_a(x) = ax} \text{ for any } x \in L,$$

satisfies the following :

$$\boldsymbol{\phi_a(x) = ax \leq x} \quad \text{or}$$

$$\boldsymbol{x \leq [x : a]}$$

where $\boldsymbol{[x : a]}$ is the upper adjoint of $\boldsymbol{\phi_a(x)}$.

Contractive Principal Mappings between Posets

- For any poset P, a mapping $f : P \to P$ is called **contractive** if **$f(x) \leq x$** for all $x \in P$.

- If f has an upper adjoint $g : P \to P$, then f is contractive if and only if **$x \leq g(x)$** for all $x \in P$.

- A mapping $f : P \to P$ between posets is called a **contractive principal mapping** if f is both contractive and principal.

THEOREM 9 :

A mapping $K : \mathrm{Idl}(\mathbb{Z}) \to \mathrm{Idl}(\mathbb{Z})$ between lattices of ideals of the ring of integers is an **injective principal mapping** if and only if there is an order isomorphism $H : \mathrm{Idl}(\mathbb{Z}) \to \mathrm{Idl}(\mathbb{Z})$ and a non-zero ideal I of $\mathbb{Z}$ such that $K = F_I \circ H$ where $F_I(J) = IJ$ for any $J \in \mathrm{Idl}(\mathbb{Z})$.

COROLLARY 10 :

A mapping $K : \mathrm{Idl}(\mathbb{Z}) \to \mathrm{Idl}(\mathbb{Z})$ is an **injective contractive principal mapping** if and only if

$K = F_I$ for some non-zero $I \in \mathrm{Idl}(\mathbb{Z})$.

Further Works

- For a given commutative ring R, determine all the (injective, contractive) principal mappings from the lattice $\mathrm{Idl}(R)$ to itself.

- Let R be a commutative ring. Characterize the ideals of R of the form $F(R)$ for some principal mapping $F : \mathrm{Idl}(R) \to \mathrm{Idl}(R)$.

  (Note: Every principal ideal of R is such an ideal.)

- If every weak principal mapping from a bounded lattice L to itself is a principal mapping, must L be modular ?

Selected References

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THANK YOU!