Is n-permutability prime?

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The lattice of interpretability types

Interpretation from variety $\mathcal{V}$ to variety $\mathcal{W}$ is

- map $i\colon \operatorname{Clo}\mathcal{V} \to \operatorname{Clo}\mathcal{W}$ mapping n-ary terms to n-ary terms, and preserving composition and projections, or
- functor $I\colon \mathcal{W} \to \mathcal{V}$ commuting with forgetful functor.

We say that $\mathcal{V}$ is interpretable in $\mathcal{W}$ ($\mathcal{V} \leq \mathcal{W}$) if there exists interpretation $i\colon \mathcal{V} \to \mathcal{W}$.

Interpretability form quasi-order. By standard technique, we can get the coresponding partial order (we factor by equi-interpretability).

The resulting order is in fact lattice, let's denote it $\mathcal{L}$ (Garcia, Taylor, The lattice of interpretability types of varieties, 1984).

Joins in $\mathcal{L}$

Join of two varieties $\mathcal{V}$ and $\mathcal{W}$ in $\mathcal{L}$ can be described as the variety $\mathcal{V} \vee \mathcal{W}$ whose operations are operations of both varieties (taken as a discreet union of operations of $\mathcal{V}$ and operations $\mathcal{W}$), and whose indentities are all identities of both varieties.

In the other worlds, we can describe algebras in $\mathcal{V} \vee \mathcal{W}$ as $(A, F \cup G)$ where $(A, F) \in \mathcal{V}$ and $(A, G) \in \mathcal{W}$.

Example (Malcev $\vee$ majority)

Let $\mathcal{V}$ be variety with single Maltsev operation $q(x, x, y) = q(y, x, x) = y$, and $\mathcal{W}$ be variety $\mathcal{W}$ with majority operation $x = m(x, x, y) = m(x, y, x)$ $= m(y, x, x)$. Then

$$p(x, y, z) = q(x, m(x, y, z), z)$$

is a Pixley term ($p(x, x, y) = p(y, x, x) = p(y, x, y) = y$) of $\mathcal{V} \vee \mathcal{W}$. On the other hand, Pixley term implies both majority, and Malcev term. So, Pixley is the join of Malcev and majority.

Malcev conditions and prime filters

All varieties satisfying certain Malcev condition form a filter in $\mathcal{L}$.

Question

Which filters obtained in this way are prime?

Not prime

- Pixley,
- Congruence distributivity $=$ CM $\vee$ SD($\wedge$),
- NU $=$ CD $\vee$ Cube (Berman, Idziak, Marković, McKenzie, Valeriote, Willard).

Prime

- Groups, cyclic terms of prime arity (Garcia, Taylor, 1984),
- CP (Tschantz, unpublished).

Malcev conditions and prime filters, cont.

Probably prime

- Congruence modularity,
- Congruence *n*-permutability,
- Satisfying non-trivial congruence identity.

Coloring of terms by variables

(Sequeira, Barto) Let X be a given set of variables, and $A \subseteq \mathrm{Eq}(X)$. We say that variety $\mathcal{V}$ is A-colorable if there is a map $c\colon F_{\mathcal{V}}(X) \to X$ such that

1. $c(x) = x$ for all $x \in X$, and
2. for every $\alpha \in A$ whenever $f \sim_{\hat{\alpha}} g$ then $c(f) \sim_{\alpha} c(g)$

where $\hat{\alpha}$ denotes congruence of the free algebra generated by α.

Example

Let $X = \{x, y, z\}$, and $A = \{xy|z, x|yz\}$, then coloring has to satisfy that

1. if $f(x, x, z) = g(x, x, z)$ then $c(f)$ and $c(g)$ are both z, or neither of them is z,
2. if $f(x, z, z) = g(x, z, z)$ then $c(f)$ and $c(g)$ are both x, or neither of them is x.

In particular, if $\mathcal{V}$ has a Malcev term q then from 1, $c(q) = z$, and from 2, $c(q) = x$. So, $\mathcal{V}$ is not A-colorable.

And, the converse is also true, i.e., variety is CP if and only if it is not A-colorable.

Coloring conditions

We say that Malcev condition $\mathcal{P}$ satisfies coloring condition A if variety $\mathcal{V}$ satisfies $\mathcal{P}$ if and only if $\mathcal{V}$ is not A-colorable.

Folowing Malcev conditions satisfy coloring condition A for respective A's

- congruence modularity for $A = \{xy|zw, xz|yw, x|y|zw\}$,
- congruence n-permutability for $A = \{x_0x_1|x_2x_3|\dots x_n, x_0|x_1x_2|x_3\dots x_n\}$,
- satisfying non-trivial congruence identity for $A = \{xy|zw, xz|yw, x|yzw\}$.

Those are exactly linear idempotent Malcev conditions that are conjectured to be prime!

Linear varieties

Theorem (Sequeira, (Barto); Bentz-Sequeira; O)

Congruence modularity, n-permutability, and satisfying non-trivial congruence identity are prime with respect to varieties axiomatized by linear equations.

Theorem

Every Malcev condition satisfying some coloring condition is prime with respect to varieties axiomatized by linear equations.

Proof.

Suppose that $\mathcal{V}$ is A-colorable ($A \subseteq \mathrm{Eq}\, X$). Then there is a structure of $\mathcal{V}$ algebra $\mathbb{X}$ on X such that all $\alpha \in A$ are congruences of $\mathbb{X}$. Just define

$$f^{\mathbb{X}}(x_0, \dots, x_n) = c(f(x_0, x_1, \dots, x_n))$$

for every basic operation f, and check that

- these operations satisfy all linear identities of $\mathcal{V}$,
- every $\alpha \in A$ is preserved by all f's.

Locally finite and idempotent case

The idea is the same, construct relational structure $(\mathbb{X}, A)$ $(A \subseteq \mathbf{SP}(\mathbb{X}))$ with common underlying sets in both varieties which at the same time contradicts $\mathcal{P}$.

Lemma

Variety $\mathcal{V}$ is n-permutable for some n if and only if it does not allow any compatible partial order which is not an antichain.

Furthermore, if $\mathcal{V}$ is idempotent, it suffices to work with bounded partial orders.

Corollary

If $\mathcal{V}$ is locally finite, idempotent, and not n-permutable then it admits

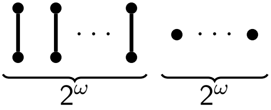
Locally finite or idempotent case

Theorem (Valeriote, Willard)

If $\mathcal{V}$ is idempotent and not n-permutable then it admits

Theorem (Barto, O)

If $\mathcal{V}$ is locally finite and not n-permutable then it admits



Corollary

If $\mathcal{V}$ and $\mathcal{W}$ are not n-permutable, and each of them is either idempotent, or locally finite then $\mathcal{V} \vee \mathcal{W}$ is not n-permutable either.

Other Malcev conditions

Theorem (McGarry, Valeriote)

Congruence modularity is prime with respect to locally finite idempotent varieties.

Theorem (Barto, O)

Congruence permutability is prime with respect to locally finite idempotent varieties.