Dualisability of a Class of Unary Algebras

Brian Schaan and Jennifer Hyndman

University of Northern British Columbia

July 2013

1. Background
    - Unary Algebras
    - $\{0,1\}$-Valued Unary Algebras with Zero
2. Our Results
    - When Rows Forms an Order Ideal
    - A More General Result
    - A Test for Non-Dualisability
3. What Next?
    - Remaining Algebras
    - Repeated Rows

Context

- A finite unary algebra is an algebra that has only unary term operations and a finite universe.

Context

- A finite unary algebra is an algebra that has only unary term operations and a finite universe.
- Unary algebras are studied extensively in universal algebra:
    - Dualisability: Clark, Davey, Pitkethly (2003); Hyndman, Willard (2000); ...
    - Bases of quasi-equations: Bestsennyi (1989); Hyndman, Casperson (2009); ...
    - Lattices of subalgebras/congruences/topologies: Nation (1974), Lampe (1974), Bordalo (1989), Kartashova (2011), ...

Advantages of Finite Unary Algebras

- A unary algebra can be represented in terms of its **Rows**.

Advantages of Finite Unary Algebras

- A unary algebra can be represented in terms of its **Rows**.

$\underline{\mathbf{P}}$	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$
0	2	1	2
1	1	2	2
2	2	2	2

Advantages of Finite Unary Algebras

- A unary algebra can be represented in terms of its **Rows**.

$\underline{\mathbf{P}}$	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$
0	2	1	2
1	1	2	2
2	2	2	2

$$\mathbf{Rows}(\underline{\mathbf{P}}) = \{\langle 2, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$$

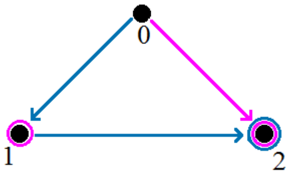
Advantages of Finite Unary Algebras

- A unary algebra can be represented in terms of its **Rows**.

$\underline{\mathbf{P}}$	f_1	f_2	f_3
0	2	1	2
1	1	2	2
2	2	2	2

$$\mathbf{Rows}(\underline{\mathbf{P}}) = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}$$

- Unary algebras are easily visualised.



Narrowing It Down

The class of all unary algebras is quite broad.

- Can narrow it down in two ways:
    - Restrict the size of the universe: Clark, Davey, and Pitkethly (2003) fully classified the dualisability of three-element unary algebras. Pitkethly (2002) extended this classification to include full and strong dualisability.
    - Impose restrictions on the term operations of the algebra: Casperson, Hyndman, Mason, Nation, and Schaan (submitted) used this approach in the context of finite bases of quasi-equations.

Casperson et al. (submitted) looked at $\{0,1\}$-valued unary algebras with zero:

- Constant function 0 that is a one-element subalgebra
- Range of all basic operations is included in $\{0,1\}$

Casperson et al. (submitted) looked at $\{0,1\}$-valued unary algebras with zero:

- Constant function 0 that is a one-element subalgebra
- Range of all basic operations is included in $\{0,1\}$

$\underline{\mathbf{M}}$	f_1	f_2	f_3
0	0	0	0
1	0	0	0
2	0	1	0
3	1	0	0

Theorem (Casperson et al. submitted)

If $\underline{\mathbf{M}}$ is a 4-element $\{0,1\}$-valued unary algbera with 0, then one of the following holds:

1. the $\leq$ on $\{0,1\}$ can be pp-defined via a formula of the form $\exists w\ x \approx f(w)\ \&\ y \approx g(w)$;
2. the graph of addition modulo 2 on $\{0,1\}$ can be pp-defined via a formula of the form $\exists w\ x \approx p(w)\ \&\ y \approx q(w)\ \&\ z \approx r(w)$;
3. the rows of $\underline{\mathbf{M}}$ form an order ideal under $0 \leq 1$.

A Place to Start

I was introduced to natural duality theory with the following question:

Question

If the rows of a $\{0, 1\}$-valued unary algebra with 0 form an order ideal under $0 \leq 1$, under what circumstances is the algebra dualisable?

Part of this question is easily answerable using this result:

Theorem (Clark, Davey, Pitkethly 2002)

Let $\underline{\mathbf{P}}$ be a finite algebra which has binary homomorphisms $\wedge$ and $\vee$ such that $\langle P; \wedge, \vee \rangle$ is a lattice. Then $\underset{\sim}{\mathbf{P}} := \langle P; \vee, \wedge, R_{2|M|}; \tau \rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{P}})$.

Part of this question is easily answerable using this result:

Theorem (Clark, Davey, Pitkethly 2002)

Let $\underline{\mathbf{P}}$ be a finite algebra which has binary homomorphisms $\wedge$ and $\vee$ such that $\langle P; \wedge, \vee\rangle$ is a lattice. Then $\underset{\sim}{\mathbf{P}} := \langle P; \vee, \wedge, R_{2|M|}; \tau\rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{P}})$.

It is straightforward to show that when the rows of an algebra form a lattice order, the conditions of this theorem are satisfied.

When Rows($\underline{\mathbf{M}}$) is not a Lattice Order?

By further narrowing the scope down to $\{0, 1\}$-valued algebras with 0 with unique rows, a pattern started to develop using a refinement of the Ghost Element Method found in Clark, Davey, and Pitkethly (2003) which states that the presence of a "ghostly element" is necessary for dualisability.

PP-Formulae

$$\exists w \ f_1(w) \approx 0 \ \& \ f_2(w) \approx f_4(w) \ \& \ x \approx f_3(w) \ \& \ y \approx f_4(w)$$

M	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{f_0}$
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**0**	**1**	**0**	**1**	**0**
3	**0**	**0**	**1**	**0**	**0**
4	**0**	**0**	**1**	**1**	**0**

PP-Formulae

$$\exists w\ f_1(w) \approx 0\ \&\ f_2(w) \approx f_4(w)\ \&\ x \approx f_3(w)\ \&\ y \approx f_4(w)$$

M	**f₁**	**f₂**	**f₃**	**f₄**	**f₀**
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**0**	**1**	**0**	**1**	**0**
3	**0**	**0**	**1**	**0**	**0**
4	**0**	**0**	**1**	**1**	**0**

PP-Formulae

$$\exists w \; f_1(w) \approx 0 \;\&\; f_2(w) \approx f_4(w) \;\&\; x \approx f_3(w) \;\&\; y \approx f_4(w)$$

M	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{f_0}$
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**0**	**1**	**0**	**1**	**0**
3	**0**	**0**	**1**	**0**	**0**
4	**0**	**0**	**1**	**1**	**0**

PP-Formulae

$$\exists w \; f_1(w) \approx 0 \;\&\; f_2(w) \approx f_4(w) \;\&\; x \approx f_3(w) \;\&\; y \approx f_4(w)$$

$\underline{\mathbf{M}}$	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{f_0}$
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**0**	**1**	**0**	**1**	**0**
3	**0**	**0**	**1**	**0**	**0**
4	**0**	**0**	**1**	**1**	**0**

This pp-formula pp-defines the relation $R = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$ on $\underline{\mathbf{M}}$.

V-Ghostable Algebras

Definition

Suppose there exist $Z \subseteq F^{\underline{\mathbf{M}}}$, distinct t and $u \in F^{\underline{\mathbf{M}}} \setminus Z$, and a collection $\{E_i\}$ of subsets of $F^{\underline{\mathbf{M}}}$ such that we can pp-define the relation $R = \{(0,0),(0,1),(1,0)\}$ via

$$\begin{aligned}\Phi: \ & \exists w[\mathop{\&}_{z\in Z} z(w) \approx 0] \\ & \& \left[\mathop{\&}_{E\in\{E_i\}} [\mathop{\&}_{d,e\in E} d(w) \approx e(w)]\right] \\ & \&[x \approx t(w)]\&[y \approx u(w)]\end{aligned}$$

such that if w_1 and w_2 witness the same element of R, then $w_1 = w_2$. Then $\underline{\mathbf{M}}$ is a **v-ghostable algebra**.

Example

$$\exists w \; [\mathop{\&}_{z \in Z} z(w) \approx 0] \& \left[\mathop{\&}_{E \in \{E_i\}} [\mathop{\&}_{d,e \in E} d(w) \approx e(w)] \right] \& [x \approx t(w)] \& [y \approx u(w)]$$

M	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{f_0}$
0	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**
2	**0**	**1**	**0**	**1**	**0**
3	**0**	**0**	**1**	**0**	**0**
4	**0**	**0**	**1**	**1**	**0**

$$\exists w \;\; f_1(w) \approx 0 \;\&\; f_2(w) \approx f_4(w) \;\&\; x \approx f_3(w) \;\&\; y \approx f_4(w)$$

Theorem 1: V-Ghosting Theorem

V-ghostable algebras are not dualisable.

Theorem 1: V-Ghosting Theorem

V-ghostable algebras are not dualisable.

Idea of Proof

Having a v-ghosting formula provides a uniform way to apply the refined Ghost Element Method.

Revisiting Order Ideals

With some work it can be shown that if $\mathbf{Rows}(\underline{\mathbf{M}})$ form an order ideal which is not a lattice order, then $\underline{\mathbf{M}}$ is v-ghostable. This gives our second result:

Revisiting Order Ideals

With some work it can be shown that if $\mathbf{Rows}(\underline{\mathbf{M}})$ form an order ideal which is not a lattice order, then $\underline{\mathbf{M}}$ is v-ghostable. This gives our second result:

Theorem 2

Let $\underline{\mathbf{M}}$ be a $\{0,1\}$-valued unary algebra with 0 with unique rows. If $\mathbf{Rows}(\underline{\mathbf{M}})$ forms an order ideal under $0 \leq 1$, then $\underline{\mathbf{M}}$ is dualisable if and only if $\mathbf{Rows}(\underline{\mathbf{M}})$ forms a lattice order.

It may not be immediately obvious whether or not an algebra is a v-ghostable algebra.

It may not be immediately obvious whether or not an algebra is a v-ghostable algebra.

$\underline{\mathbf{M}}_0$	f_1	f_2	f_3
0	0	0	0
1	0	1	0
2	1	0	0

$\underline{\mathbf{M}}$	f_1	f_2	f_3	f_4	f_0
0	0	0	0	0	0
1	0	0	1	1	0
2	1	0	1	0	0
3	1	1	0	0	0
4	1	1	0	1	0

It may not be immediately obvious whether or not an algebra is a v-ghostable algebra.

$\mathbf{M}_0$	f_1	f_2	f_3
0	0	0	0
1	0	1	0
2	1	0	0

$\mathbf{M}$	f_1	f_2	f_3	f_4	f_0
0	0	0	0	0	0
1	0	0	1	1	0
2	1	0	1	0	0
3	1	1	0	0	0
4	1	1	0	1	0

$\exists w \ x \approx f_1(w) \ \& \ y \approx f_2(w)$

Two Term Reducts

Definition

By a **two term reduct** of an algebra $\underline{\mathbf{M}} = \langle M; F^{\underline{\mathbf{M}}} \rangle$ we mean an algebra $\underline{\mathbf{N}} = \langle M; F^{\underline{\mathbf{N}}} \rangle$ where $F^{\underline{\mathbf{N}}}$ consists of exactly two of the functions from $F^{\underline{\mathbf{M}}}$.

Two Term Reducts

Definition

By a **two term reduct** of an algebra $\underline{\mathbf{M}} = \langle M; F^{\underline{\mathbf{M}}} \rangle$ we mean an algebra $\underline{\mathbf{N}} = \langle M; F^{\underline{\mathbf{N}}} \rangle$ where $F^{\underline{\mathbf{N}}}$ consists of exactly two of the functions from $F^{\underline{\mathbf{M}}}$.

$\underline{\mathbf{M}}$	f_1	f_2	f_3	f_4	f_0
0	0	0	0	0	0
1	0	0	1	1	0
2	1	0	1	0	0
3	1	1	0	0	0
4	1	1	0	1	0

Two Term Reducts

Definition

By a **two term reduct** of an algebra $\underline{\mathbf{M}} = \langle M; F^{\underline{\mathbf{M}}}\rangle$ we mean an algebra $\underline{\mathbf{N}} = \langle M; F^{\underline{\mathbf{N}}}\rangle$ where $F^{\underline{\mathbf{N}}}$ consists of exactly two of the functions from $F^{\underline{\mathbf{M}}}$.

$\underline{\mathbf{M}}$	f_1	f_2	f_3	f_4	f_0
0	0	**0**	**0**	0	0
1	0	**0**	**1**	1	0
2	1	**0**	**1**	0	0
3	1	**1**	**0**	0	0
4	1	**1**	**0**	1	0

$\underline{\mathbf{N}}$	f_2	f_3	f_0
0	0	0	0
1	0	1	0
2	0	1	0
3	1	0	0
4	1	0	0

V-Order Reductable

Definition

If $\underline{\mathbf{M}}$ has a two term reduct $\underline{\mathbf{N}}$ such that $\mathbf{Rows}(\underline{\mathbf{N}}) = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$ such that the row $\langle 0,0\rangle$ is uniquely witnessed (in $\underline{\mathbf{N}}$), then we say that $\underline{\mathbf{M}}$ is **v-order reductable**.

V-Order Reductable

Definition

If $\underline{\mathbf{M}}$ has a two term reduct $\underline{\mathbf{N}}$ such that $\mathbf{Rows}(\underline{\mathbf{N}}) = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$ such that the row $\langle 0,0\rangle$ is uniquely witnessed (in $\underline{\mathbf{N}}$), then we say that $\underline{\mathbf{M}}$ is **v-order reductable**.

$\underline{\mathbf{M}}$	f_1	f_2	f_3	f_4	f_0
0	0	**0**	**0**	0	0
1	0	**0**	**1**	1	0
2	1	**0**	**1**	0	0
3	1	**1**	**0**	0	0
4	1	**1**	**0**	1	0

$\underline{\mathbf{N}}$	f_2	f_3	f_0
0	0	0	0
1	0	1	0
2	0	1	0
3	1	0	0
4	1	0	0

V-Order Reductable Theorem

Every v-order reductable algebra is v-ghostable.

V-Order Reductable Theorem

Every v-order reductable algebra is v-ghostable.

Corollary

Every v-order reductable algebra is not dualisable.

V-Order Reductable Theorem

Every v-order reductable algebra is v-ghostable.

Corollary

Every v-order reductable algebra is not dualisable.

Idea of Proof

We use a double induction on the number of occurrences of the rows $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ in the reduct to build up v-ghosting formulae.

What Next?

There are $\{0, 1\}$-valued unary algebras with 0 with unique rows to which these results do not apply.

What Next?

There are $\{0,1\}$-valued unary algebras with 0 with unique rows to which these results do not apply.

$\underline{\mathbf{M}}_1$	f_1	f_2	f_3	f_0
0	0	0	0	0
1	0	1	1	0
2	1	0	1	0
3	1	1	0	0

$\underline{\mathbf{M}}_2$	f_1	f_2	f_3	f_0
0	0	0	0	0
1	0	0	1	0
2	0	1	1	0
3	1	0	1	0

What Next?

There are $\{0,1\}$-valued unary algebras with 0 with unique rows to which these results do not apply.

$\underline{\mathbf{M}}_1$	f_1	f_2	f_3	f_0
0	0	0	0	0
1	0	1	1	0
2	1	0	1	0
3	1	1	0	0

$\underline{\mathbf{M}}_2$	f_1	f_2	f_3	f_0
0	0	0	0	0
1	0	0	1	0
2	0	1	1	0
3	1	0	1	0

Neither is dualisable. Ross Willard assisted us with proving this for $\underline{\mathbf{M}}_1$. The proof for $\underline{\mathbf{M}}_2$ utilizes Pitkethly's (2010) result that if a finite unary algebra is dualisable, it is dualisable via a finite set of relations.

Repeated Rows

All of our results only apply when the rows of $\underline{\mathbf{M}}$ are not repeated. The results do not appear to generalize intuitively.

Repeated Rows

All of our results only apply when the rows of $\underline{\mathbf{M}}$ are not repeated. The results do not appear to generalize intuitively.

$\underline{\mathbf{M}}_3$	f_1	f_2	f_0
0	0	0	0
1	0	1	0
2	1	0	0

$\underline{\mathbf{M}}_4$	f_1	f_2	f_0
0	0	0	0
1	0	0	0
2	0	1	0
3	1	0	0

Repeated Rows

All of our results only apply when the rows of $\underline{\mathbf{M}}$ are not repeated. The results do not appear to generalize intuitively.

$\underline{\mathbf{M}}_3$	f_1	f_2	f_0
0	0	0	0
1	0	1	0
2	1	0	0

$\underline{\mathbf{M}}_4$	f_1	f_2	f_0
0	0	0	0
1	0	0	0
2	0	1	0
3	1	0	0

Not Dualisable

Repeated Rows

All of our results only apply when the rows of $\underline{\mathbf{M}}$ are not repeated. The results do not appear to generalize intuitively.

$\underline{\mathbf{M}}_3$	f_1	f_2	f_0
0	0	0	0
1	0	1	0
2	1	0	0

Not Dualisable

$\underline{\mathbf{M}}_4$	f_1	f_2	f_0
0	0	0	0
1	0	0	0
2	0	1	0
3	1	0	0

Dualisable

Acknowledgements

Many thanks to Jennifer Hyndman, David Casperson, Ross Willard, and Jesse Mason for their contributions to this research. I gratefully acknowledge the University of Northern British Columbia and the Natural Sciences and Engineering Research Council of Canada for their funding of my studies.

References

Bestsennyi, I.P. (Sept-Oct 1989). *Quasiidentities of finite unary algebras*, Algebra i Logika **28**, no. 5, pp. 493-512.

Bordalo, G. (1989). *A duality between unary algebras and their subuniverse lattices*, Portugaliae Mathematica **46**, no. 4, pp. 431-439.

Casperson, D. & Hyndman, J. (2009). *Primitive positive formulas preventing a finite basis of quasi-equations*, International Journal of Algebra and Computation **19**, no. 7, pp. 925-935.

Casperson, D., Hyndman, J., Mason, J., Nation, J.B. & Schaan, B. *Existence of finite bases for quasi-equations of unary algebras with 0*, submitted.

Clark, D.M. & Davey, B.A. (1998). *Natural dualities for the working algebraist* (pp. 41-51). Cambridge, UK: Cambridge University Press.

Clark, D.M., Davey, B.A. & Pitkethly, J. (2002). *Binary homomorphisms and natural dualities*, Journal of Pure and Applied Algebra **169**, pp. 1-28.

Clark, D.M., Davey, B.A. & Pitkethly, J. (2003). *The complexity of dualisability: Three-element unary algebras*, International Journal of Algebra and Computation **13**, no. 3, pp. 361-391.

Hyndman, J. & Willard, R. (2000). *An algebra that is dualizable but not fully dualizable*, Journal of Pure and Applied Algebra **151**, pp. 3142.

Kartashova, A.V. (2011). *On congruence lattices and topology lattices of unary algebras*, Chebyshevskiĭ Sbornik **12**, no. 2(38), pp. 27-33.

Lampe, W.A. (1974). *Subalgebra lattices of unary algebras and an axiom of choice*, Colloquium Mathematicum **30**, pp. 41-55.

Nation, J.B. (1974). *Congruence lattices of relatively free unary algebras*, Algebra Universalis **4**, pp. 132.

Pitkethly, J. (2010). *The finite type problem for unary algberas*, Acta Scientiarum Mathematicarum **76**, no. 3-4, pp. 359-370.

Pitkethly, J. (2002). *Strong and full dualisability: three-element unary algebras*, Journal of the Australian Mathematical Society **73**, no. 2, pp. 187-219.

Thank You