

# Finitely decidable varieties admitting type 1 are residually finite

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# The Finite Decidability Problem

Let  $\mathcal{V}$  be a variety (usually locally finite) in a finite language. We say  $\mathcal{V}$  is *decidable* if its first-order theory is, and *finitely decidable* if the theory of  $\mathcal{V}_{\text{fin}}$  is decidable.

Finitely decidable varieties

McKenzie & Smedberg

The Problem

Bounding SIs in  $\mathcal{V}$

$\text{Rad}(S)$  is strongly abelian

$\mathcal{V}$  is residually finite

Open Problems

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Decidable and finitely decidable varieties are rare and structurally constrained. For example,

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Fact (Willard, Valeriote, Idziak, Jeong, McKenzie, S., . . .)

- ▶ If  $\mathbf{A}$  has any congruence covers of the lattice or semilattice types, or

then every variety containing  $\mathbf{A}$  is finitely undecidable.

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- ▶ If  $\mathbf{A}$  has any congruence covers of the lattice or semilattice types, or
- ▶ If any boolean- or affine-type minimal sets in  $\mathbf{A}$  have nonempty tails, or

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- ▶ *If  $\mathbf{A}$  is a subdirectly irreducible finite algebra with two incomparable nonabelian congruences,*

*then every variety containing  $\mathbf{A}$  is finitely undecidable.*

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## Fact (continued)

*If  $\mathbf{A}$  is a finite algebra*

- ▶ *and  $\mathbf{A}$  has a solvable congruence which is nonabelian, or*

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- ▶ *and  $\mathbf{A}$  has a solvable congruence which is nonabelian, or*
- ▶  *$\mathbf{A}$  is subdirectly irreducible with boolean monolith and also has a cover of type 1 or 2, or*

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- ▶  *$\mathbf{A}$  is SI with type 2 monolith and has also a cover of type 1, or vice versa,*

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These facts (and many of a similar nature) were established for modular varieties in the 90s (see [Idziak 1997]). The results for nonmodular varieties are in most cases new.

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# Bounding Subdirect Irreducibles in $\mathcal{V}$

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## Theorem (McKenzie, S.)

*Let  $\mathcal{K}$  be a finite set of finite algebras, and suppose  $\mathcal{V} = \text{HSP}(\mathcal{K})$  is finitely decidable. Then there is a finite bound on the cardinalities of SI algebras in  $\mathcal{V}$ .*

The Problem

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Type 3 and 2

Type 1

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Using familiar methods from the congruence-modular case, we show that

- ▶ every SI with boolean-type monolith belongs to  $\text{HS}(\mathcal{K})$ ;

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Using familiar methods from the congruence-modular case, we show that

- ▶ every SI with boolean-type monolith belongs to  $\text{HS}(\mathcal{K})$ ;
- ▶ there is a bound ( $\sim$  quadruply exponential) on the affine-type SIs.

# Bounding unary-type SIs in $\mathcal{V}$

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So let  $\mathbf{S} \in \mathcal{V}$  have monolith  $\perp \prec^1 \mu$ .

## Lemma

$\text{Rad}_u(\mathbf{S})$  is comparable to all congruences of  $\mathbf{S}$ .

## Lemma

$\text{Rad}_u(\mathbf{S})$  is meet-irreducible.

Each of these is proved by contradiction: supposing the respective lemma were false, we construct a (relatively straightforward) interpretation of some finitely undecidable class into  $\text{HSP}(\mathbf{S})$ .

The Problem

Bounding SIs in  $\mathcal{V}$

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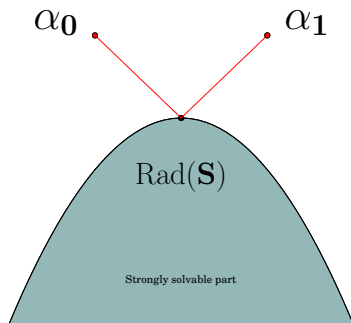
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Open Problems

# Meet-irreducibility of the solvable radical

Goal: to semantically interpret a structure of the form  $\langle I; E_0, E_1 \rangle$  (where the  $E_j$  are disjoint equivalence relations) into subpowers of  $\mathbf{S}$ .



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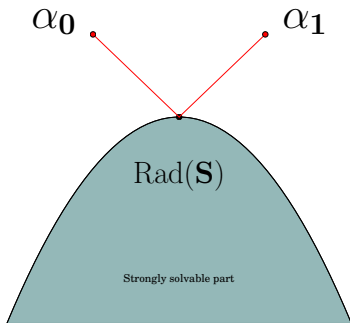
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Let  $\{0_j, 1_j\}$  be  $(\text{Rad}_u(\mathbf{S}), \alpha_j)$ -minimal sets. Let  $\mathbf{B} \leq \mathbf{S}'$  consist of all  $\mathbf{x}$  which are  $\alpha_1$ -constant on  $E_1$ -blocks and vice versa.



The Problem

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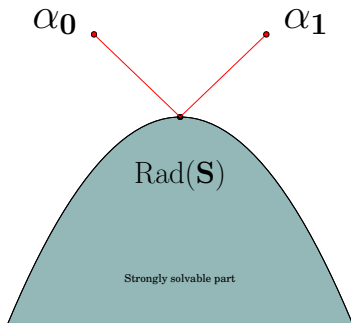
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Using a failure of  $C(\mu, \{0_j, 1_j\}; \perp_S)$ , and some tricks from tame congruence theory,

we reconstruct the original structure  $\langle I; E_0, E_1 \rangle$  in a first-order way from  $\mathbf{B}$ .



The Problem

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Open Problems

Since  $\text{Rad}_u(\mathbf{S})$  is meet-irreducible, we know that its index cannot exceed the maximum size of a boolean-type SI in  $\mathcal{V}$ .

## Theorem

$\text{Rad}_u(\mathbf{S})$  is strongly abelian.

## Proof.



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## Theorem

$\text{Rad}_u(\mathbf{S})$  is strongly abelian.

## Proof.

Long! □

Takeaway idea: Subalgebra generation (and congruence generation) can frequently be proven to be “sparse” in some useful sense, when the generators are chosen so that they are almost constant modulo a strongly abelian congruence (such as the monolith).

# Sparse subalgebra generation: Example I

Finitely decidable  
varieties

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Suppose  $C(\theta_0, \mu|_U; \perp)$  holds in our subdirectly irreducible algebra, but  $C(\theta_1, \mu|_U; \perp)$  does not, where  $\theta_0 \prec \theta_1$  are strongly solvable.

The Problem

Bounding SIs in  $V$

Rad(S) is strongly  
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Sparsity of Sg and Cg

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Open Problems

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See Lemma 3.1 in our paper for more about [this example](#).



# Sparse subalgebra generation: Example I

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Choose a witnessing package

$$t(a_0, \vec{b}_0) = t(a_0, \vec{b}_1)$$

but

$$t(a_1, \vec{b}_0) \neq t(a_1, \vec{b}_1)$$

where  $t$  takes values in some  $\perp, \mu$ -minimal set.

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## Example I, continued

Now suppose  $\mathbb{G} = \langle V, E \rangle$  is a graph we want to interpret into  $\text{HSP}(\mathbf{S})$ . Generate  $\mathbf{D} \leq \mathbf{S}^{V \sqcup \{\infty\}}$

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## Example I, continued

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Now suppose  $\mathbb{G} = \langle V, E \rangle$  is a graph we want to interpret into  $\text{HSP}(\mathbf{S})$ . Generate  $\mathbf{D} \leq \mathbf{S}^{V \sqcup \{\infty\}}$  using all the diagonal elements, plus all elements of the form

$$a_{1|\{v, \infty\}} \oplus a_{0|\text{else}} \quad (v \in V)$$

$$a_{1|\{v, w, \infty\}} \oplus a_{0|\text{else}} \quad (v \xrightarrow{E} w)$$

plus one extra element  $m_{0|V} \oplus m_{1|\{\infty\}}$  ( $m_0 \neq m_1$  belong to some  $(\perp, \mu)$ -trace).

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Now suppose  $\mathbb{G} = \langle V, E \rangle$  is a graph we want to interpret into  $\text{HSP}(\mathbf{S})$ . Generate  $\mathbf{D} \leq \mathbf{S}^{V \sqcup \{\infty\}}$  using all the diagonal elements, plus all elements of the form

$$\begin{aligned} a_{1|\{v, \infty\}} \oplus a_{0|\text{else}} & \quad (v \in V) \\ a_{1|\{v, w, \infty\}} \oplus a_{0|\text{else}} & \quad (v \xrightarrow{E} w) \end{aligned}$$

plus one extra element  $m_{0|V} \oplus m_{1|\{\infty\}}$  ( $m_0 \neq m_1$  belong to some  $(\perp, \mu)$ -trace).

### Key Claim

Every point in  $D \cap U$  attains at most two values (mod  $\theta_0$ ), and does so precisely in the pattern of one of the generators (i.e. one of these values occurs at a vertex and infinity, or at the endpoints of an edge and at infinity).

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Open Problems



# Sparse subalgebra generation: Example II

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Open Problems

Fix a  $(\perp, \mu)$ -minimal set  $U$ , and say we are working to semantically interpret a graph  $\langle V, E \rangle$  into a power of  $\mathbf{S}$ . Let  $I = \{v^+, v^- : v \in V\}$ . Define a subalgebra

$$\Delta \subseteq \mathbf{B} \leq \mathbf{S}^I$$

with generators those  $\mathbf{x} \in U^I$  such that for some  $v \in V$ ,

$$\begin{cases} x^{v^+} \equiv_{\mu} x^{v^-} \\ x^{w^+} = x^{w^-} \equiv_{\mu} x^{v^+} \end{cases} \quad \text{for all other } w \in V$$

## Example II continued

Finitely decidable varieties

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### Claim

*$B \cap U^I$  consists of just the generators and no more.*

The Problem

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## Example II continued

Finitely decidable varieties

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### Claim

$B \cap U^I$  consists of just the generators and no more.

Proof: write an arbitrary element  $\mathbf{y} \in U^I$  as a product  $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_k)$  of generators, where  $\mathbf{f} = f^I$  for some polynomial operation  $f : \mathbf{S} \rightarrow U$ . Let  $C_j$  be the  $\mu$ -class where  $\mathbf{x}_j$  lives; then on  $C_1 \times \dots \times C_k$ ,  $f$  is essentially unary; say it depends on  $\mathbf{x}_1$ , which has its spike at  $v_0 \in V$ . Then  $y^{v_0^+} \equiv_{\mu} y^{v_0^-}$ , and for all  $w \neq v_0$ ,

$$x_1^{w^+} = x_1^{w^-} \quad \text{and} \quad x_j^{w^+} \equiv_{\mu} x_j^{w^-}$$

so that

$$y^{w^+} = f(x_1^{w^+}, \dots, x_k^{w^+}) = f(x_1^{w^-}, \dots, x_k^{w^-}) = y^{w^-}$$

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# Sparse congruence generation: Example III

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Open Problems

Assume that  $\sigma = \text{Rad}_\mu(\mathbf{S})$  is abelian over  $\mu$  but not over  $\perp$ , and let  $\mathbb{G} = \langle V, E \rangle$  be a graph. Fix the index set  $I = V \times \{+, -\}$ , and let  $\mathbf{D} \leq \mathbf{S}'$  be the subalgebra consisting of all  $\sigma$ -constant points.

---

See Lemma 3.6 in our paper for all the hypotheses of this example. 

## Sparse congruence generation: Example III

Finitely decidable varieties

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The Problem

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Next, choose a  $(\perp, \mu)$ -subtrace  $\{m_0, m_1\}$ , and let  $\Theta$  be the congruence on  $\mathbf{D}$  generated by identifying all pairs

$$m_1|_{v^+} \oplus m_0|_{\text{else}} \equiv m_1|_{v^-} \oplus m_0|_{\text{else}} \quad (v \in V)$$

$$m_1|_{v^+, w^+} \oplus m_0|_{\text{else}} \equiv m_1|_{v^-, w^-} \oplus m_0|_{\text{else}} \quad (v \stackrel{E}{\sim} w)$$

---

See Lemma 3.6 in our paper for all the hypotheses of this example. 

## Key Claim

When restricted to a minimal set,  $\Theta$  contains blocks of cardinality 1 and 2 only, and if  $\mathbf{x} \equiv_{\Theta} \mathbf{y}$  then the set of coordinates where they differ is either empty, or  $\{v^+, v^-\}$  for some  $v \in V$ , or  $\{v^+, w^+, v^-, w^-\}$  for some  $v \stackrel{E}{\sim} w$ .

# Bounding $\text{Rad}_u(\mathbf{S})$ -blocks

Finitely decidable varieties

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Say  $\text{Rad}_u(\mathbf{S})$  has index  $\ell$  and some fixed monolith pair  $c \neq d$ .

Since  $\text{Rad}_u(\mathbf{S})$  is strongly abelian,

The Problem

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Bounding  $\text{Rad}(\mathbf{S})$ -blocks

Open Problems

## Lemma

*For any polynomial  $t(v_0, \vec{v}_1, \dots, \vec{v}_\ell)$ , there exist subsets of each variable set  $\vec{v}_i$ , of size no more than  $\log |\mathbf{F}_V(2 + \ell)|$ , such that for all  $\text{Rad}_u(\mathbf{S})$ -blocks  $B_1, \dots, B_\ell$ , the mapping*

$$A \times \vec{B}_1 \times \dots \times \vec{B}_\ell \rightarrow A$$

*induced by  $t$  depends only on  $v_0$  and the indicated subsets.*

Because of the Lemma, terms  $f(v_0) = t(v_0, \vec{s})$  of bounded arity suffice to send exactly one of any unequal elements  $x_1 \neq x_2$  to  $c$ .

## Bounding $\text{Rad}_u(\mathbf{S})$ -blocks, II

Finitely decidable varieties

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The Problem

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Bounding  $\text{Rad}(\mathbf{S})$ -blocks

Open Problems

Consider a fixed  $\text{Rad}_u(\mathbf{S})$ -block  $B$ , and to each  $b \in B$  associate the set of terms  $t(v_0, v_1, \dots, v_k)$ , with  $k$  bounded as described in the last slide, such that for some  $p_1, \dots, p_k$  from the appropriate  $\text{Rad}_u(\mathbf{S})$ -blocks,  $t(b, \vec{p}) = c$ .

### Claim

*This is an injective map from  $B$  to subsets of  $\mathbf{F}_{\mathcal{V}}(1+k)$*

For if not, we get a failure of the strong term condition

$$c = t(b_1, \vec{p}_1) = t(b_2, \vec{p}_2) \text{ but } t(b_2, \vec{p}_1) \neq c$$

This contradiction completes the proof.



## Problem

*Do finitely decidable, locally finite varieties have definable principal congruences? Definable principal subcongruences? Definable principal solvable congruences?*

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Finitely decidable varieties

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## Problem

*In a finite algebra  $\mathbf{A}$  in a finitely decidable variety, must every congruence permute with  $\text{Rad}(\mathbf{A})$ ? With  $\text{Rad}_u(\mathbf{A})$ ?*

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## Problem

*In all known cases, the set of finitely refutable sentences of a finitely generated variety is either decidable or Turing-complete. Do there exist varieties where this set has an intermediate complexity class?*

# Thank you!