Topological systems versus attachment relations

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Outline

Topological systems and attachment relations

- There exist two notions related to topology: *topological system* of S. Vickers and *attachment relation* of C. Guido.
- Topological systems were introduced in 1989 as a common framework for both topological spaces and their underlying algebraic structures – frames or locales.
- Attachment relations were introduced in 2009 as a fuzzification of the relation "$\in$" ("element of") between points and sets.

Variety-based approach

- In 2010, the notions of *variety-based topological system* and *attachment relation* were introduced, thereby suggesting the problem of comparing systems and attachments.
- Both concepts are given by an indexed family of homomorphisms of algebras of some variety, where topological systems use an indexing over a set, and attachment relations put the algebraic structure of the variety in question on it.

Topological systems versus attachment relations

- This talk shows an embedding of the category of attachment relations into the category of topological systems.
- We also construct a functor in the opposite direction, which is an embedding under certain restrictions on its domain.
- The two functors make a link between the spatialization and localification procedures for systems and attachments, where the attachment localification procedure is defined for this occasion.
- We construct, additionally, another functor from systems to attachments, and provide a natural transformation between the two system-to-attachment ways.

Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda\in\Lambda}$ be a class of cardinal numbers.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda\in\Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda\in\Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Definition 3

Given a variety $\mathbf{A}$, a *reduct* of $\mathbf{A}$ is a pair $(|-|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$ (every primitive operation of $\mathbf{B}$ is a primitive operation of $\mathbf{A}$), whereas $\mathbf{A} \xrightarrow{|-|} \mathbf{B}$ is a concrete functor.

Variety-based powerset operators

Theorem 4

Given a variety $\mathbf{A}$, *every subcategory* $\mathbf{S}$ *of* $\mathbf{A}^{op}$ *induces the functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\leftarrow}} \mathbf{A}^{op}$, *which is defined by* $(X_1, A_1) \xrightarrow{(f,\varphi)^{\leftarrow}} (X_2, A_2) = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$, *where* $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Example 5

1. If $\mathbf{A} = \mathbf{CBAlg}$ (complete Boolean algebras) and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, then $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ is the classical preimage operator.
2. If $\mathbf{A} = \mathbf{Frm}$ (frames), then $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\leftarrow}} (\mathbf{Loc} := \mathbf{Frm}^{op})$ (locales) is the powerset operator of S. E. Rodabaugh.

Variety-based topological spaces

Definition 6

Given a variety **A**, a subcategory **S** of $\mathbf{A}^{op}$ and a reduct **B** of **A**, $\mathbf{S_B}$-**Top** is the category, concrete over the category **Set** $\times$ **S**, which comprises the following data.

Objects: triples (X, A, τ) ($\mathbf{S_B}$-*topological spaces* or $\mathbf{S_B}$-*spaces*), where (X, A) is a **Set** $\times$ **S**-object, and τ ($\mathbf{S_B}$-*topology* on (X, A)) is a **B**-subalgebra of $|A^X|$.

Morphisms: $(X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)$ are **Set** $\times$ **S**-morphisms $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$ with the property that $(f, \varphi)^{\leftarrow}(\alpha_2) \in \tau_1$ for every $\alpha_2 \in \tau_2$ ($\mathbf{S_B}$-*continuity*).

Examples

Example 7

1. If $\mathbf{A} = \mathbf{CBAlg}$, $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, and $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{S_B}$-**Top** is the category **Top** of topological spaces.
2. If $\mathbf{A} = \mathbf{B} = \mathbf{Frm}$, then $\mathbf{S_B}$-**Top** is the category **S**-**Top** of lattice-valued topological spaces of S. E. Rodabaugh.

Variety-based topological systems

Definition 8

Given a variety **A**, a subcategory **S** of $\mathbf{A}^{op}$ and a reduct **B** of **A**, $\mathbf{S_B}$-**TopSys** is the category, concrete over the category $\mathbf{Set} \times \mathbf{B}^{op} \times \mathbf{S}$, which comprises the following data.

Objects: tuples $D = (\operatorname{pt} D, \Omega\, D, \Sigma\, D, \models)$ (*$\mathbf{S_B}$-topological systems* or *$\mathbf{S_B}$-systems*), where $(\operatorname{pt} D, \Omega\, D, \Sigma\, D)$ is a $\mathbf{Set} \times \mathbf{B}^{op} \times \mathbf{S}$-object, and $\operatorname{pt} D \times \Omega\, D \xrightarrow{\models} \Sigma\, D$ is a map (*$\Sigma\, D$-satisfaction relation*) such that $\Omega\, D \xrightarrow{\models(x,-)} |\Sigma\, D|$ is a **B**-homomorphism for every $x \in \operatorname{pt} D$.

Morphisms: $D_1 \xrightarrow{f=(\operatorname{pt} f,(\Omega\, f)^{op},(\Sigma\, f)^{op})} D_2$ are $\mathbf{Set} \times \mathbf{B}^{op} \times \mathbf{S}$-morphisms $(\operatorname{pt} D_1, \Omega\, D_1, \Sigma\, D_1) \xrightarrow{f} (\operatorname{pt} D_2, \Omega\, D_2, \Sigma\, D_2)$ such that $\Sigma\, f \circ \models_2 (\operatorname{pt} f(x), b) = \models_1 (x, \Omega\, f(b))$ for every $x \in \operatorname{pt} D_1$, $b \in \Omega\, D_2$.

Examples

Example 9

1. If $\mathbf{A} = \mathbf{CBAlg}$, $\mathbf{B} = \mathbf{Frm}$, and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, then $\mathbf{S_B}$-**TopSys** is the category **TopSys** of topological systems of S. Vickers.
2. If $\mathbf{A} = \mathbf{B} = \mathbf{Frm}$, and $\mathbf{S} = \mathbf{Loc}$, then $\mathbf{S_B}$-**TopSys** is the category of lattice-valued topological systems of S. E. Rodabaugh.

System spatialization procedure

Theorem 10

1. *There exists the full embedding* $\mathbf{S_B\text{-}Top} \xhookrightarrow{\mathrm{E}_{TS}} \mathbf{S_B\text{-}TopSys}$, *which is defined by* $\mathrm{E}_{TS}((X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)) = (X_1, \tau_1, A_1, \models_1) \xrightarrow{(f,((f,\varphi)^{\leftarrow}|^{\tau_1}_{\tau_2})^{op},\varphi)} (X_2, \tau_2, A_2, \models_2)$, *where* $\models_i(x, \alpha) = \alpha(x)$.
2. *There exists the functor* $\mathbf{S_B\text{-}TopSys} \xrightarrow{\mathrm{Spat}_{TS}} \mathbf{S_B\text{-}Top}$, *which is defined by* $\mathrm{Spat}_{TS}(D_1 \xrightarrow{f} D_2) = (\mathrm{pt}\, D_1, \Sigma\, D_1, \tau_1) \xrightarrow{(\mathrm{pt}\, f,(\Sigma\, f)^{op})} (\mathrm{pt}\, D_2, \Sigma\, D_2, \tau_2)$, *where* $\tau_i = \{\models_i(-, b) \mid b \in \Omega\, D_i\}$.
3. Spat_{TS} *is a right-adjoint-left-inverse to* E_{TS}.

System localification procedure

Theorem 11

1. *Every object A of the category* $\mathbf{S}$ *gives rise to an embedding* $\mathbf{B}^{op} \xhookrightarrow{\mathrm{E}^{A}_{TS}} \mathbf{S_B}\text{-}\mathbf{TopSys}$ *defined by* $\mathrm{E}^{A}_{TS}(B_1 \xrightarrow{\varphi} B_2) = (Pt_A(B_1), B_1, A, \models_1) \xrightarrow{((\varphi^{op})^{\leftarrow}_{A}, \varphi, 1_A)} (Pt_A(B_2), B_2, A, \models_2)$, *where* $Pt_A(B_i) = \mathbf{B}(B_i, |A|)$ *and* $\models_i(p, b) = p(b)$. E^{A}_{TS} *is full iff* $\mathbf{S}(A, A) = \{1_A\}$.
2. *There exists the functor* $\mathbf{S_B}\text{-}\mathbf{TopSys} \xrightarrow{\mathrm{Loc}^{\Omega}_{TS}} \mathbf{B}^{op}$ *defined by* $\mathrm{Loc}^{\Omega}_{TS}(D_1 \xrightarrow{f} D_2) = \Omega\, D_1 \xrightarrow{(\Omega\, f)^{op}} \Omega\, D_2$.
3. $\mathrm{Loc}^{\Omega}_{TS}$ *is a left inverse to* E^{A}_{TS}. *If* $\mathbf{S} = \{A \xrightarrow{1_A} A\}$, *then* $\mathrm{Loc}^{\Omega}_{TS}$ *is a left adjoint to* E^{A}_{TS}.

Variety-based attachment relations

Definition 12

Given a variety **A**, a subcategory **S** of $\mathbf{A}^{op}$, a reduct **B** of **A**, and a functor $\mathbf{B}^{op} \xrightarrow{(-)^*} \mathbf{Set}$ such that $B^* = |B|$, $\mathbf{S}_{\mathbf{B}}$-**Att** is the category, concrete over the category $\mathbf{B}^{op} \times \mathbf{S}$, comprising the following data.

Objects: triples $F = (\Omega F, \Sigma F, \Vdash)$ ($\mathbf{S}_{\mathbf{B}}$*-attachments*), where $(\Omega F, \Sigma F)$ is a $\mathbf{B}^{op} \times \mathbf{S}$-object, and $\Omega F \xrightarrow{\Vdash} \mathbf{B}(\Omega F, |\Sigma F|)$ is a map (ΣF*-attachment relation*).

Morphisms: $F_1 \xrightarrow{f = ((\Omega f)^{op}, (\Sigma f)^{op})} F_2$ are $\mathbf{B}^{op} \times \mathbf{S}$-morphisms $(\Omega F_1, \Sigma F_1) \xrightarrow{f} (\Omega F_2, \Sigma F_2)$ such that $\Sigma f \circ \Vdash_2((\Omega f)^{op*}(b_1))(b_2) = (\Vdash_1(b_1))(\Omega f(b_2))$ for every $b_1 \in \Omega F_1$, $b_2 \in \Omega F_2$ ($\mathbf{S}_{\mathbf{B}}$*-continuity*).

Example

Example 13

If $\mathbf{A} = \mathbf{CBAlg}$, $\mathbf{B} = \mathbf{Frm}$, and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, then $\mathbf{S}$-$\mathbf{Att}$ is the category of attachment relations of C. Guido.

Attachment spatialization procedure

Theorem 14

1. *There exists the functor* $\mathbf{S_B}$-$\mathbf{Top} \xrightarrow{\mathrm{E}_{AT}} \mathbf{S_B}$-$\mathbf{Att}$, *which is given by* $\mathrm{E}_{AT}((X_1, A_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, A_2, \tau_2)) = (\tau_1, A_1^{X_1}, \Vdash_1) \xrightarrow{(((f,\varphi)^{\leftarrow}|_{\tau_2}^{\tau_1})^{op},((f,\varphi)^{\leftarrow})^{op})} (\tau_2, A_2^{X_2}, \Vdash_2)$, *where* $(\Vdash_i(\alpha))(\alpha') = \alpha'$. E_{AT} *is injective on objects, but, in general, is not faithful and, therefore, fails to be an embedding.*
2. *There exists the functor* $\mathbf{S_B}$-$\mathbf{Att} \xrightarrow{\mathrm{Spat}_{AT}} \mathbf{S_B}$-$\mathbf{Top}$, *which is defined by the formula* $\mathrm{Spat}_{AT}(F_1 \xrightarrow{f} F_2) = (|\,\Omega\, F_1|, \Sigma\, F_1, \tau_1) \xrightarrow{((\Omega\, f)^{op*},(\Sigma\, f)^{op})} (|\,\Omega\, F_2|, \Sigma\, F_2, \tau_2)$, *where* $\tau_i = \{(\Vdash_i(-))(b) \mid b \in \Omega\, F_i\}$.
3. Spat_{AT} *is not a left inverse to* E_{AT}.

Remarks

While comparing the spatialization procedures for systems and attachments, one could emphasize three important differences.

- The functor E_{AT} is not an embedding.
- Spat_{AT} is not a left inverse to E_{AT}.
- Since the definition of the category $\mathbf{S_B}$-**Att** involves a functor $\mathbf{B}^{op} \xrightarrow{(-)^*} \mathbf{Set}$, whose nature, in general, is unclear, we are unable to determine whether Spat_{AT} is a right adjoint to E_{AT}.

Attachment localification procedure

Theorem 15

1. *There exists the full embedding* $\mathbf{S} \xhookrightarrow{\mathrm{E}^{\mathbf{S}}_{AT}} \mathbf{S_B}\text{-}\mathbf{Att}$, *which is defined by* $\mathrm{E}^{\mathbf{S}}_{AT}(A_1 \xrightarrow{\varphi} A_2) = (|A_1|, A_1, \Vdash_1) \xrightarrow{(\varphi,\varphi)} (|A_2|, A_2, \Vdash_2)$, *where* $\Vdash_i(a) = 1_{A_i}$.
2. *There exists the functor* $\mathbf{S_B}\text{-}\mathbf{Att} \xrightarrow{\mathrm{Loc}^{\Sigma}_{AT}} \mathbf{S}$, *which is given by* $\mathrm{Loc}^{\Sigma}_{AT}(F_1 \xrightarrow{f} F_2) = \Sigma F_1 \xrightarrow{(\Sigma f)^{op}} \Sigma F_2$.
3. $\mathrm{Loc}^{\Sigma}_{AT}$ *is a left inverse to* $\mathrm{E}^{\mathbf{S}}_{AT}$. *In general,* $\mathrm{Loc}^{\Sigma}_{AT}$ *is not a left adjoint to* $\mathrm{E}^{\mathbf{S}}_{AT}$.

Embedding attachments into systems

Theorem 16

There exists the embedding $\mathbf{S_B\text{-}Att} \overset{\mathrm{H}}{\hookrightarrow} \mathbf{S_B\text{-}TopSys}$ *given by*

$$\mathrm{H}(F_1 \xrightarrow{f} F_2) =$$

$$(|\,\Omega F_1|, \Omega F_1, \Sigma F_1, \models_1) \xrightarrow{((\Omega f)^{op*}, (\Omega f)^{op}, (\Sigma f)^{op})} (|\,\Omega F_2|, \Omega F_2, \Sigma F_2, \models_2),$$

where $\models_i(b, b') = (\Vdash_i(b))(b')$.

From systems to attachments

Theorem 17

There exists the functor $\mathbf{S_B}$**-TopSys** $\xrightarrow{\mathsf{K}}$ $\mathbf{S_B}$**-Att** *given by*

$$\mathsf{K}(D_1 \xrightarrow{f} D_2) =$$

$$(\Omega\, D_1, (\Sigma\, D_1)^{\mathrm{pt}\, D_1}, \Vdash_1) \xrightarrow{((\Omega\, f)^{op}, ((\mathrm{pt}\, f, (\Sigma\, f)^{op})^{\leftarrow})^{op})} (\Omega\, D_2, (\Sigma\, D_2)^{\mathrm{pt}\, D_2}, \Vdash_2),$$

where $((\Vdash_i(b))(b'))(x) = \models_i(x, b')$ *for every* $b, b' \in \Omega\, D_i, x \in \mathrm{pt}\, D_i$*.*
The functor K *is injective on objects.*

Embedding systems into attachments

Definition 18

$\mathbf{S}_{\mathbf{B}}$-$\mathbf{TopSys}^{*}_{\Sigma}$ is the non-full subcategory of $\mathbf{S}_{\mathbf{B}}$-$\mathbf{TopSys}$, comprising the following data.

Objects: $\mathbf{S}_{\mathbf{B}}$-systems D such that $\operatorname{pt} D$ is non-empty, and $\Sigma\, D$ has at least two elements.

Morphisms: $\mathbf{S}_{\mathbf{B}}$-system morphisms $D_1 \xrightarrow{f} D_2$ such that the map $\Sigma\, D_2 \xrightarrow{\Sigma\, f} \Sigma\, D_1$ is injective.

Theorem 19

The restriction of the functor $\mathbf{S}_{\mathbf{B}}$-$\mathbf{TopSys} \xrightarrow{\mathsf{K}} \mathbf{S}_{\mathbf{B}}$-$\mathbf{Att}$ *to* $\mathbf{S}_{\mathbf{B}}$-$\mathbf{TopSys}^{*}_{\Sigma}$ *is an embedding.*

From systems to attachments again

Theorem 20

1. *There exists the functor* $\mathbf{S_B}$-$\mathbf{TopSys} \xrightarrow{\mathsf{L}} \mathbf{S_B}$-$\mathbf{Att}$ *given by*

$$\mathsf{L}(D_1 \xrightarrow{f} D_2) = ((\Omega\, D_1)^{\mathrm{pt}\, D_1}, (\Sigma\, D_1)^{\mathrm{pt}\, D_1}, \Vdash_1) \xrightarrow{(((\mathrm{pt}\, f, (\Omega\, f)^{op})^{\leftarrow})^{op}, ((\mathrm{pt}\, f, (\Sigma\, f)^{op})^{\leftarrow})^{op})} ((\Omega\, D_2)^{\mathrm{pt}\, D_2}, (\Sigma\, D_2)^{\mathrm{pt}\, D_2}, \Vdash_2),$$

where $((\Vdash_i(\alpha))(\alpha'))(x) = \models_i(x, \alpha'(x))$ *for every* $\alpha, \alpha' \in (\Omega\, D_i)^{\mathrm{pt}\, D_i}$, $x \in \mathrm{pt}\, D_i$. *The functor* L *is injective on objects.*

2. *The restriction of* L *to* $\mathbf{S_B}$-$\mathbf{TopSys}^{*}_{\Sigma}$ *is an embedding.*

Embedding systems into attachments again

Definition 21

$\mathbf{S_B}$-$\mathbf{TopSys}^{*}_{\Omega}$ is the non-full subcategory of $\mathbf{S_B}$-$\mathbf{TopSys}$, comprising the following data.

Objects: $\mathbf{S_B}$-systems D such that $\mathrm{pt}\, D$ is non-empty, and $\Omega\, D$ has at least two elements.

Morphisms: $\mathbf{S_B}$-system morphisms $D_1 \xrightarrow{f} D_2$ such that the map $\Omega\, D_2 \xrightarrow{\Omega\, f} \Omega\, D_1$ is injective.

Theorem 22

The restriction of the functor $\mathbf{S_B}$-$\mathbf{TopSys} \xrightarrow{\mathsf{L}} \mathbf{S_B}$-$\mathbf{Att}$ *to* $\mathbf{S_B}$-$\mathbf{TopSys}^{*}_{\Omega}$ *is an embedding.*

Relationships between two embeddings

Theorem 23

There exists a natural transformation

$$\mathbf{S_B}\text{-}\mathbf{TopSys} \underset{K}{\overset{L}{\Downarrow \eta}} \mathbf{S_B}\text{-}\mathbf{Att},$$

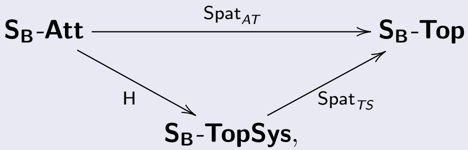
which is defined by the formula $\mathsf{L}\,D \xrightarrow{\eta_D} \mathsf{K}\,D = ((\Omega D)^{\mathrm{pt}\,D}, (\Sigma D)^{\mathrm{pt}\,D}, \Vdash_{\mathsf{L}}) \xrightarrow{(([1_{\Omega\,D}]_{\mathrm{pt}\,D})^{op}, 1_{(\Sigma\,D)^{\mathrm{pt}\,D}})} (\Omega D, (\Sigma D)^{\mathrm{pt}\,D}, \Vdash_{\mathsf{K}})$.

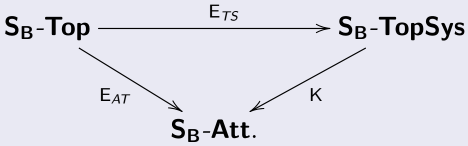
η provides a connection between *variety-based point-set* (involving powersets $(\Omega\,D)^{\mathrm{pt}\,D}$ or $(\Sigma\,D)^{\mathrm{pt}\,D}$) and *variety-based pointless* (involving just a **B**-algebra $\Omega\,D$ or an **A**-algebra $\Sigma\,D$) topologies.

The functors H, K and the spatialization procedure

Theorem 24

The following triangles commute:

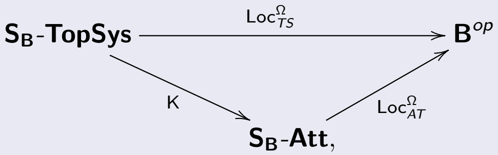


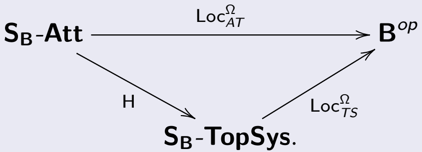


The functors H, K and the localification procedure

Theorem 25

The following triangles commute:





Another system localification procedure

Theorem 26

1. *There exists the embedding*

$$\mathbf{S} \xhookrightarrow{\mathrm{E}^{\mathbf{S}}_{TS}} \mathbf{S_B}\text{-}\mathbf{TopSys} \;=\; \mathbf{S} \xhookrightarrow{\mathrm{E}^{\mathbf{S}}_{AT}} \mathbf{S_B}\text{-}\mathbf{Att} \xhookrightarrow{\mathrm{H}} \mathbf{S_B}\text{-}\mathbf{TopSys}$$

defined by $\mathrm{E}^{\mathbf{S}}_{TS}(A_1 \xrightarrow{\varphi} A_2) = (|A_1|, |A_1|, A_1, \models_1) \xrightarrow{(\varphi^*,\varphi,\varphi)} (|A_2|, |A_2|, A_2, \models_2)$, *where* $\models_i(a, -) = 1_{A_i}$.

2. *There exists the functor* $\mathbf{S_B}\text{-}\mathbf{TopSys} \xrightarrow{\mathrm{Loc}^{\Sigma}_{TS}} \mathbf{S}$, *which is given by* $\mathrm{Loc}^{\Sigma}_{TS}(D_1 \xrightarrow{f} D_2) = \Sigma\, D_1 \xrightarrow{(\Sigma\, f)^{op}} \Sigma\, D_2$.

3. $\mathrm{E}^{\mathbf{S}}_{TS}$ *is a right inverse to* $\mathrm{Loc}^{\Sigma}_{TS}$, *and, in general, is not full.*

Final remarks

- This talk considered functorial relationships between the categories of (variety-based) topological systems of S. Vickers and attachment relations of C.Guido.
- The talk showed an embedding of the category of attachment relations into the category of topological systems, suggesting a meta-mathematical conclusion that the idea of topological system is more general than the idea of attachment relation.
- We also constructed two functors in the opposite direction, which are embeddings under some restrictions on their domains only, and showed a link between them, which (ultimately) provided two morphisms of variety-based topological theories.

Open problem

This talk constructed an embedding $\mathbf{S_B}\text{-}\mathbf{Att} \overset{\mathrm{H}}{\hookrightarrow} \mathbf{S_B}\text{-}\mathbf{TopSys}$,

and, additionally, two functors $\mathbf{S_B}\text{-}\mathbf{TopSys} \underset{\mathrm{L}}{\overset{\mathrm{K}}{\rightrightarrows}} \mathbf{S_B}\text{-}\mathbf{Att}$.

Problem 27

Is it true that the category $\mathbf{S_B}\text{-}\mathbf{Att}$ *is isomorphic to a (co)reflective subcategory of the category* $\mathbf{S_B}\text{-}\mathbf{TopSys}$*? More precisely, is it true that the embedding* H *has a left or a right adjoint?*

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Thank you for your attention!