Rings with kernel inclusion

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The theory of *relation algebras* was Tarski's attempt to model the algebra of binary relations on a set X – subsets of $X \times X$.

Many operations are defined:

- relational composition $\cdot$ (generalises function composition)
- all the usual set operations: $\cup$, $\cap$, $'$, plus "nullaries" 0 (the empty relation) and U (the universal relation)
- relational converse $*$ (reverse all pairs)
- the diagonal relation 1 (= identity function)

Tarski came up with an apparently comprehensive set of laws for relation algebra.

But this class of “relation algebras” still fails to capture actual algebras of relations!

There are relation algebras that are not concrete algebras of relations on any set X.

The class of concrete relation algebras is an infinitely based variety (needs an infinite set of equational axioms to specify it).

So there is no “finite axiomatization” of concrete relation algebras.

However, when one restricts to various reducts, nice axiomatizations often exist.

For example, the algebras of binary relations under composition are precisely semigroups!

But there are many other cases that work (eg composition plus intersection).

And there are further operations one can consider, such as...

domain $D(f)$: the restriction of the diagonal 1 to the domain of f, or

range $R(f)$, anaologusly defined in terms of range.

Often, the concrete algebras of relations give rise to an abstract class of algebras with no finite axiomatization.

How about the set $P(X)$ of all partial functions $X \to X$?

These are functions whose domain may not be all of X, ie “functional binary relations”.

Possible operations now include composition (still associative), but also some but not all set-theoretic operations, as well as some others.

In fact things seem to work better for partial functions (although fewer set operations make sense for them).

But a number of different natural *relations* on function (and relation) semigroups have also been axiomatized; many of them quasiorders.

Example: the axioms for domain inclusion (= set-theoretic containment of domains) $\lhd$ in semigroups of partial functions are as follows:

- $\lhd$ is a quasiorder
- $st \lhd s$
- $s \lhd t$ implies $us \lhd ut$

(Here we read compositions left to right.)

This means any semigroup with such a quasiorder can be faithfully represented as a semigroup of partial functions in such a way that $\triangleleft$ is represented as domain inclusion.

In fact the same axioms apply to domain inclusion for semigroups of binary relations.

A given semigroup will in general accommodate many such domain inclusion relations.

These correspond to different ways of representing it as functions/relations.

There are many other relations of interest:

- range inclusion (also makes sense for transformartions)
- kernel inclusion (ditto)
- set-theoretic containment $\subseteq$
- the relation that two partial functions agree where both are defined (not a quasi-order)
- the range of f is contained in the domain of g
- . . . and others.

There is interest in extending these results to structure-preserving maps of various kinds.

An obvious possible case: associative rings.

Every ring is embeddable in the endomorphism ring of an abelian group (its own if it has identity).

Here the endomorphism operations are the pointwise ones plus composition.

One might ask: what additional operations or relations could make sense for endomorphisms?

No further operations spring to mind, but a number of relations do.

One of the key notions for groups and rings is the notion of *kernel*.

For $f \in End(A)$,

$$ker(f) = \{a \in A \mid f(a) = 0\}.$$

Likewise, *image* is fundamental:

$$Im(f) = \{f(a) \mid a \in A\}.$$

For $f, g \in End(A)$, we can define $f \lesssim g$ if $ker(f) \subseteq ker(g)$.

Call this the *kernel inclusion quasiorder*.

This quasiorder is quite expressive.

For example, let $f, g, h, k \in End(A)$.

Then the implication $\forall a \in A: \; f(a) = g(a) \Rightarrow h(a) = k(a)$ is equivalent to

$$f - g \lesssim h - k.$$

It is not hard to write down a few laws satisfied by any kernel inclusion quasi-order $\lesssim$ on R.

Of course, it is a quasiorder. Otherwise, it is easy enough to see that the following laws all hold.

- $s \lesssim -s$
- $s \lesssim t,\ s \lesssim u \ \Rightarrow\ s \lesssim t + u$
- $s \lesssim st$
- $s \lesssim t \Rightarrow us + ns \lesssim ut + nt$ (n an integer)

We also need the law

- $0 \lesssim r \;\Rightarrow\; r = 0$.

(This last law may be omitted, and we obtain a more general kind of quasiorder that makes sense for arbitrary right R-modules, not just faithful ones.)

If R contains the identity function, the fourth law can be replaced by

- $s \lesssim t \Rightarrow us \lesssim ut$.

Theorem: The above axioms are complete for rings of endomorphisms equipped with $\lesssim$.

This means that these laws hold in any ring of endomorphisms (subring of $End(A)$) equipped with kernel inclusion. (Soundness.)

But it also means that any ring equipped with such a quasiorder can be represented as endomorphisms of an abelian group equipped with kernel inclusion.

Sketch of proof: (Unital case: the more general case is an easy variant using Dorroh extensions.)

Soundness is a straightforward verification.

Conversely, consider the right R-module obtained by factoring out the right ideal I of R which is up-closed under $\lesssim$ ($i \in I$ and $i \lesssim a$ implies $a \in I$).

Then R acts on R/I in a way that is order-preserving:

$a \lesssim b$ implies $ker(a) \subseteq ker(b)$ as kernel inclusion, but maybe not conversely.

Take the direct product of all the R/I and let R act on this; now $\lesssim$ can be shown to be faithfully represented.

(If $r \not\lesssim s$, then $s \notin I_r = \{x \in R \mid r \lesssim x\}$, so when R acts on R/I_r, $(r : 0) \not\subseteq (s : 0)$. Now extend to the direct product.)

So R equipped with $\lesssim$ is now faithfully represented in $End(A)$. $\Box$

Call those quasiorders on R satisfying the above axioms *ker-orders*.

It is possible to characterise the ker-orders on R in terms of certain families of right ideals of R.

A given ker-order on a ring R determines the set of all up-closed right ideals of R with respect to it; in turn this family determines the ker-order.

Such families $\mathcal{I}$ may be described without reference to ker-orders.

They satisfy

- $\mathcal{I}$ is closed under intersections
- $\mathcal{I}$ is closed under unions (where this makes sense)
- $R \in \mathcal{I}$
- if $I \in \mathcal{I}$, so is $(r + n : I) = \{a \in R \mid ra + na \in I\}$.

The associated ker-order is given by $a \lesssim b$ if $a \in I$ implies $b \in I$ for all $I \in \mathcal{I}$.

It satisfies faithfulness iff $\bigcap \mathcal{I} = \{0\}$.

The set of all ker-orders on a fixed ring forms a complete lattice in which meet is intersection.

It is anti-isomorphic to the lattice of sets of right ideals satisfying the above properties.

Those satisfying the faithfulness condition constitute a sublattice.

This lattice need not be modular.

Other relations of interest on $End(A)$.

Consider the following relation:

“the image of f is contained in the kernel of g”.

This is already expressible in ring theory:

$$fg = 0.$$

Similarly, we can express the relation “the image of f intersects trivially with the kernel of g” in terms of $\lesssim$:

$$fg \lesssim f.$$

But image inclusion apparently cannot be reduced like this.

Axioms for image inclusion in rings would be of interest.

So would axioms for image and kernel inclusion together.

This talk is based on the following paper:

“Rings with kernel inclusion quasiorder”,

to appear in *Algebra Universalis*.